Social Learning with Payoff Externality: Why Some Learn While Others Do Not

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Abstract

This paper studies a model of social learning with externalities and possibly with limited observability of past actions. Bayesian agents sequentially choose an action each, after observing the actions of a subset of preceding agents. These actions and a private signal, conditionally independent across agents, supply information about the common state of the world. Each agent’s payoff depends on the state and the actions of the subset of preceding agents (neighbors).

With complete observability of past actions, I show that social learning occurs even in the presence of externality. Social learning also occurs with limited observability but in the absence of externality. However with both limited observability of the history of actions and action externalities, social learning is bounded away from full information. When different agents have randomly determined observability, even a small chance of having limited observability can hamper social learning. These results stand in contrast to the existing literature on herding and social learning. I also show that when agents know the extent of observability of previous agents, agents with better observability learn while the rest do not.

KEYWORDS: Social Learning, Information Aggregation, Externality.
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1 Introduction

The social learning literature studies settings where agents act sequentially, after observing the actions of previous agents, as well as a private signal about an underlying state that determines the payoff to various actions. The central question is: Can the society aggregate these individual signals and correctly infer the state? In other words, do agents who act later learn more about the underlying hidden state from actions of previous agents and converge towards the optimal decision? The literature shows that social learning happens under mild conditions. These models assume that agents observe the entire past and each agent’s payoff depends on his/her actions and on the state. The motivation for the current paper is that in many environments both assumptions above are invalid, i.e. agents have limited observation of past agents’ actions, and payoff externalities are present. My work shows that, ceteris paribus, social learning may fail in such settings.

For concreteness, start with a model with an infinite sequence of firms with overlapping generations, each choosing a location. The location need not be a physical location but can be a choice of market, or marketing strategy, or production technique. Firms live for two periods, locate when it is young and produce in the same location the next period. Each firm sees only the previous firm’s decision. The state of the world is 0 if there is negative externality across firms; for example the firms might have exploited a common resource to the point of depletion, or polluted a resource excessively. In state 1, the externality is positive; we may interpret this as arising from ready access to transportation network that developed in response to the earlier firm and in having easier access to skilled personnel or an existing customer base. In state 0 each firm wants to avoid co-locating with its predecessor firm to avoid the negative externalities, while it does the opposite in state 1. However states are not known. Each firm gets a continuously distributed private signal in \([0, 1]\) about the state. A high signal is highly suggestive of a good state, i.e. 1; opposite for low. Thus this setting naturally incorporates both limited observability and externalities.

Consider the decision problem of firms. The first firm locates randomly. Suppose it
chooses the city rather than the suburbs. Firm 2 learns nothing from firm 1’s decision because 1 chose randomly. Firm 2 gets a signal and if it is high enough it hopes to reap positive externalities and locates in the city. When firm 3 comes along, it sees that firm 2 is located in the city, but doesn’t know where firm 1 was, because the latter is no longer operating. There are two scenarios: 1) firm 1 chose the city and firm 2 got a high signal and followed suit; 2) firm 1 chose the suburbs and firm 2 got a low signal and opted for the city. With a symmetric specification of various components of the model it can be shown that both these scenarios are equally likely. Firm 3 doesn’t learn anything above and beyond the information contained in its private signal. In this setting the result is stark — there is no social learning.

Two notable papers on Bayesian social learning are Bikhchandani et al. (1992) and Smith and Sørensen (2000). These show that this information aggregation can often lead to wrong conclusions: when economic agents act sequentially after observing the actions of all previous agents, it is possible to have no learning. This can be due to information cascades as explained in Bikhchandani et al. (1992). Or, when there are multiple agent types, convoluted learning may happen as in Smith and Sørensen (2000).

Two key modeling assumptions are used in these papers. First, agents have perfect observability in that every agent (individual) can observe the actions of all agents preceding her. And second, the actions of the preceding agents don’t affect the payoff (or utility) function of an agent. In other words, there is no externality of the action of an individual on the payoff of subsequent agents. However, in practice, individuals are often part of complex social networks, which not only is their main source of information, but often also imposes an externality on their payoff. And most often, these networks do not include all preceding agents. Further, the strength and form of externality can also vary from simple to complex.

I start with a canonical sequential learning problem, except that as in Acemoglu et al.

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1 Suppose we were to allow each firm to observe the last two actions; then each firm can infer the motives for the preceding firm’s action; this would restore social learning. So would removing the externality.
instead of full observation of past actions, the agent can have limited observability and she only observes the actions of a few agents preceding her. I call the collection of these agents as her neighborhood. I further introduce the presence of externality of those past actions within the neighborhood on the payoff (and hence action) of the agent. More specifically, a large number of agents sequentially choose between two actions. An underlying state and the actions of preceding agents who are in an agent’s neighborhood, determine the payoffs from these two actions. Each agent receives a private noisy signal about the state of the world and observes the actions of preceding agents in her neighborhood. All agents have identical preference structure. The learning process is characterized by two features: (i) the private signal structure, which determines how informative the signals are, and (ii) the neighborhood structure which determines the actions that the agent observes as well the nature of payoff externality. The neighborhood structure can be deterministic, in that every agent has the same level of limited observability about the past, or it can be stochastic where the observed actions of an agent (her neighborhood) are determined by a probability distribution.

In this setting, I consider different possible payoff externalities and characterize the types of neighborhood structure that can or cannot lead to information aggregation and social learning. I show that, in the absence of payoff externality, even limited observability will lead to social learning. Next, if payoff externality is present, but agents observe all past actions, again social learning occurs. But if the observability of agents is too limited, in the presence of payoff externality, information accumulation is not possible. Even when there is a very small chance of having limited observability (under a structure of stochastic observability), social learning breaks down. However, when agents know the type of observability preceding agents had, the subgroup with better observability is able to accumulate information, whereas the rest fail to do so.

Figure 1 illustrates the position of this paper in sequential social learning literature. In the context of our canonical model with unbounded private beliefs social learning always occurs except in the setting with both payoff externality and limited observability.
The rest of the paper is organized as follows. The next section gives a brief overview of the literature and clearly demarcates the contribution of my paper. Section 3 introduces the model and provides illustrative examples. Section 4 provides the main results of the paper. I first compare the learning dynamics between the baseline case (that of no externality) and that of externality with very limited observability. Next, I show that a better observability leads to social learning and discuss the speed of learning in such a case. Then, I discuss the situation of a stochastic observability. Finally, Section 5 concludes with a sketch of areas for future research.

1.1 Some Other Examples

The key feature of the example discussed above is that it is the state of the world that determines what effect actions of previous agents will have on an agent's payoff structure. We can find this motif in a variety of other situations as well.
(A) Many sectors of the economy (often the ones with high entry costs) exhibit increasing returns to scale. Whereas there are other sectors exhibiting diminishing returns. A sequence of investors each chooses between two possible assets. An individual investor doesn’t know whether these assets suffer from increasing or diminishing returns. Increasing returns means it is profitable for her to invest in the asset in which most of her predecessors have invested. Whereas, diminishing returns means she should invest in the asset in which the preceding investors haven’t invested as much. Obviously, she will be interested in trying to learn about whether she is facing increasing or diminishing returns. She has a private signal which is correlated with the type of return and she can also observe the investment decisions of previous investors. But she doesn’t always observe the investment decisions of all preceding investors (as some of them might have moved out of the market). Can this sequence of investors accumulate their individual signals and learn about the type of returns they are facing?

(B) Consider an overlapping generation model of investment. An individual invests when young and enjoys the return from that investment when old. There are two possible assets to invest in, both of which suffer from an aggregate risk and an idiosyncratic risk. A higher level of investment in an asset lowers the idiosyncratic risk. But the aggregate risk is determined by nature and is unknown to the agents. Each investor observes a private signal about the level of aggregate risk before investing. In case of default, the agent with seniority (the agent investing earlier) gets her money back while the investor in the next generation looses her money.

Then, if the aggregate risk is low, idiosyncratic risk is a bigger concern for an investor and so she will invest in the asset in which the last generation has already invested. Whereas, if the aggregate risk is high, the investor will prefer to invest in an asset which gives her seniority (the one which the last generation hasn’t invested in) so that in case of a default she will get her money back. The investor wants to learn about the level of aggregate risk from the actions taking by previous genera-
tions and their private signals. But observing all previous actions might not always be possible as preceding agents have already received their returns and have moved out of the market. In such a situation, is information accumulation possible? Does this sequence of investors learn about the level of aggregate risk in the economy?

(C) Consider a farmland which is handed down generations. Only two possible crops can be grown on the farm, rice or lentils. For each generation, the right crop to plant depends on two factors. First, the quality of soil, which is the state of the world, and second, what was grown by the last generation. If the quality of the soil is good, it’s better to continue growing what the last generation did as there can be skills specific to a crop which can be passed down generations. However, if the quality of the soil is bad, it cannot support the same crop for two generations so it’s better to grow what the last generation didn’t grow. Clearly, there is a payoff externality across generations. Further, agents in each generation recall the actions of the previous generations (or two). So the question to ask is whether there will be any information accumulation about the quality of the soil over successive generations.

A general case of this externality can be seen where people are trying to learn not only about an underlying state from the action of other individuals, but also, in the process, learn skills related to those actions. To take any action requires some level of skill or technological know-how. And this requires some personal cost. However, observing other people take these actions helps to lower the personal costs of acquiring the related skills. This can be seen not only in agriculture, as was the case in the last example, but also in a variety of other areas.

(D) Such externalities can also be observed in the market for movies and video games. I want to watch a movie that suits my tastes (high quality), but my friends’ choice of movies also affects my preference. So my final action depends on a combined effect of my belief about the quality of the movie on one hand, and the number of friends who have watched this movie on the other hand. A similar process also works in the video games market. More specifically, for multi-player video games
my decision to buy a game depends not only on the inherent quality of the game but also on its popularity among my friends.

2 Related Literature

The vast social learning literature can be divided roughly into two groups based on whether learning is Bayesian or non-Bayesian. Typically, Bayesian models focus on learning by observing past actions, while non-Bayesian learning models focus on learning by communicating the beliefs about the state. Bikhchandani et al. (1992) and Banerjee (1992) are generally considered the ones who started the literature on learning when individuals observe past actions and use Bayesian updating. Smith and Sørensen (2000) provide a more comprehensive analysis of this environment allowing continuum of signals, multiple agent types, bounded and unbounded private beliefs. These papers typically focus on (i) the special case (in terms of my model) of the neighborhood of an agent including all preceding agents (perfect observability) so that she can observe all past actions and (ii) assume no externality on her payoff from past actions.

The problem of limited observability has been dealt with in a few papers like Banerjee and Fudenberg (2004), Acemoglu et al. (2011) and Jehiel and Newman (2011). More specifically, Jehiel and Newman (2011) deals with the implication of limited observability on learning in a principal-agent setting. The principal observes contracts offered by other principals where agents have cheated and tries to modify her own contract to close those loopholes. In this setting limited observability (observing concurrent principals only) leads to a non-stationary distribution of contracts among principals changing between contracts that deter cheating and contracts that encourage cheating. On the other hand, Banerjee and Fudenberg (2004) and Acemoglu et al. (2011) deal with sequential learning of an underlying common state of the world (as in my model). In both these papers, each agent observes actions of only a subset of preceding agents. In Banerjee and Fudenberg (2004), each agent from a continuum observe a representative sample of the overall popu-
Asymptotic learning is achieved under mild assumptions as long as the sample size is at least two. On the other hand, Acemoglu et al. (2011) consider a model very similar to mine with a countable number of agents acting sequentially. The neighborhood of an agent, in this paper, is generated stochastically and the realized neighborhood is a subset of past agents whose actions are observed. Asymptotic learning in this case requires a slightly stronger condition that there be no finite influential group. A set of agents $S$ is influential if any agent’s neighborhood is a subset of $S$.

A crucial difference between all the above papers on one hand, and my paper on the other hand is the existence of payoff externality and how that distorts learning. In the environment of my paper, the action of agents in someone’s neighborhood not only influence her belief about the state of the world but also her payoff for any given state of the world. I view this as a better approximation of neighborhoods.

The other branch of the social learning literature is characterized by non-Bayesian learning, where agents use reasonable rules of thumb. Within this literature, the papers most closely related to mine are Bala and Goyal (1998) and Golub and Jackson (2010). Both study non-Bayesian learning over an arbitrary, connected social network. My paper can be viewed as extending the results of Bala and Goyal (1998) to a situation with Bayesian learning. But the crucial difference between these and my paper is again the presence of payoff externality. Golub and Jackson (2010) provide a characterization result about which individuals in the social network will be influential and show that learning (in this case it is the development of consensus opinion) occurs only in the absence of an influential group that is connected with a large number of agents.

### 3 The Model

There are two payoff-relevant states of the world: $\Theta = \{0, 1\}$. There is a common prior with $Pr(\theta = 0) = Pr(\theta = 1) = 1/2$. A countably infinite set of agents, indexed by $n \in \mathbb{N}$,
each make a single decision, sequentially. Agent \( n \) receives a private signal \( s_n \in S \equiv [0, 1] \) about the state of the world. Conditional on the state of the world \( \theta \in \Theta \), the signals are independently generated according to the probability measure \( F_\theta \) on \([0, 1] \). This forms the signal structure of the model given by \( (F_0, F_1) \). By assumption \( F_0 \) and \( F_1 \) are absolutely continuous with respect to each other, which implies that no signal is fully revealing about the underlying state of the world. However, \( F_0 \) and \( F_1 \) are not identical so that some signals are informative: the Radon-Nikodym derivative \( \frac{dF_0}{dF_1}(s_n) \neq 1 \) in a subset of \( S \) of positive measure.

As in Acemoglu et al. (2011), agent \( n \) does not necessarily observe all past actions. Instead, she only observes the actions of a subset of preceding agents, who are in her neighborhood \( \Lambda(n) \). So for each agent \( n \), her neighborhood \( \Lambda(n) \subset \{1, 2, \ldots, n-1\} \). The collection \( \{\Lambda(n)\}_{n \in \mathbb{N}} \) forms the realized network structure of the model.

Each agent \( n \) takes a binary decision \( a_n \in \{0, 1\} \). However, in contrast to the existing literature, the payoff of agent \( n \) not only depends on her own action and the underlying state of the world but also on the actions of a subset of agents in her neighborhood. In state \( \theta \in \Theta \), agent \( n \) earns a payoff \( u^\theta(a_n|a_j, j \in \Lambda(n)) \) from action \( a_n \in \{0, 1\} \) when her neighborhood is \( \Lambda(n) \). As she doesn’t know the state of the world, she maximizes her expected payoff.

### 3.1 Payoffs and Externality

For any given state \( \theta \) and a realized neighborhood \( \Lambda(n) \), the payoff-relevant information (due to externality), for any agent \( n \), can be represented as follows

\[
I(n) = \{(j, a_j) : j \in \Lambda(n), a_j \in \{0, 1\}\},
\]

where \( \{\Lambda(n)\}_{n \in \mathbb{N}} \) is a given network structure. The set of all possible \( I(n) \) allowed under a network structure is represented as \( \mathcal{I}(n) \).

The baseline payoff structure of this paper is the canonical form in Bayesian social learn-
ing literature. I denote this as **No Externality** (NES)\(^2\).

\[
\begin{align*}
    u^0(a_n) &= \begin{cases} 
    1 & a_n = 0 \\ 
    0 & a_n = 1 
    \end{cases} \\
    u^1(a_n) &= \begin{cases} 
    0 & a_n = 0 \\ 
    1 & a_n = 1, 
    \end{cases}
\end{align*}
\]

where \(u^\theta(a_n)\) is the payoff from taking action \(a_n\) when the underlying state is \(\theta\). This is a case of agents trying to match their action with the state, with no regard for what actions previous agents took.

The payoff structure with externality, on the other hand is denoted as **Strong Externality with Symmetric Payoff** (SES). It represents the situation where the state of the world determines the complementarity or the substitutability of action of the preceding agent on the action of the current agent. This means that if the state of the world is 1, the best option for an agent is to match the action of the preceding agent. Whereas, if the state is 0, the best option for the agent is to take the action opposite to what the last agent took.

The following describes that structure.

If \(a_{n-1} = 0\), then

\[
\begin{align*}
    u^0(a_n) &= \begin{cases} 
    0 & a_n = 0 \\ 
    1 & a_n = 1 
    \end{cases} \\
    u^1(a_n) &= \begin{cases} 
    1 & a_n = 0 \\ 
    0 & a_n = 1, 
    \end{cases}
\end{align*}
\]

Otherwise, if \(a_{n-1} = 1\), then

\[
\begin{align*}
    u^0(a_n) &= \begin{cases} 
    1 & a_n = 0 \\ 
    0 & a_n = 1 
    \end{cases} \\
    u^1(a_n) &= \begin{cases} 
    0 & a_n = 0 \\ 
    1 & a_n = 1, 
    \end{cases}
\end{align*}
\]

where \(a_0 = 1\).

\(^2\)These and other payoff structures are simplified versions of general payoff structures discussed in Appendix A.
3.2 Signal Structure and Private Beliefs

Given the signal structure \((F_0, F_1)\) and the network structure \(\{\Lambda(n)\}_{n \in \mathbb{N}}\), any agent \(n\) takes her optimum action \(a_n\) based on her posterior belief \(r_n = Pr(\theta = 1|s_n, I(n))\) about the state \(\theta\) (where \(s_n\) is her private signal); \(r_n\) can be broken down into two components: her private belief, \(p_n = Pr(\theta = 1|s_n)\), and her neighborhood belief, \(q_n = Pr(\theta = 1|I(n))\), and following Bayes’ rule, the relationship is

\[
\frac{1 - r_n}{r_n} = \left(\frac{1 - p_n}{p_n}\right) \left(\frac{1 - q_n}{q_n}\right).
\]

Next, the private belief, again from applying Bayes’ rule, satisfies

\[
\frac{1 - p_n}{p_n} = \frac{dF_0}{dF_1}(s_n),
\]

where \(\frac{dF_0}{dF_1}(s_n)\) is the Radon-Nikodym derivative. If \(F_\theta\)’s have density \(f_\theta\), then the Radon-Nikodym derivative can be represented as \(\frac{f_0(s_n)}{f_1(s_n)}\). In addition, if there is some \(s_n \in S\), for which both measures have atoms, then \(\frac{dF_0}{dF_1}(s_n) = \frac{F_0(s_n)}{F_1(s_n)}\).

Private beliefs are said to be unbounded if the support of the private beliefs is \([0, 1]\).

Since \(p_n\) is identically distributed for all agents (because the private signals are identically distributed), both the support and the conditional distribution of private beliefs is identical for all agents.

I represent the conditional distribution of private belief given by the underlying state by \(G_\theta(p) = Pr(p_n < p|\theta), \forall \theta \in \Theta\). The pair \((G_0, G_1)\) is a private belief distribution pair if there exists a signal structure \((F_0, F_1)\) such that the conditional distribution of private beliefs is given by \((G_0, G_1)\).

3.2.1 Symmetric Private Belief Distribution

To get some intuition about the working of the model, henceforth I employ a version of private belief distribution, as in Smith and Sorensen (2008), with tractable closed-form results. This distribution is as follows.

\[
G_0(p) = 2p - p^2, \quad G_1(p) = p^2,
\]
with the corresponding density functions
\[ g_0(p) = 2 - 2p, \quad g_1(p) = 2p. \]

The key property which these distributions follow is Symmetry.

![Graph showing distribution and density functions](image)

**Figure 2: Private Belief Distribution**

**Definition 3.1.** Private belief distributions satisfy the symmetry property iff
\[ G_0(p) = 1 - G_1(1 - p). \]

To motivate this class of signals and private distributions, consider the discrete case of \( s_n \in \{L, H\} \) with \( Pr(s_n = L|\theta = L) = l = 1 - Pr(s_n = H|\theta = L) \), so that the private belief \( p_n \) can either be \( l \) or \((1 - l)\). This means that \( g_0(p) = g_1(1 - p) \), which implies that the symmetry property holds.

### 3.3 Equilibrium and Learning

Given the payoff and information structure mentioned in section 3.1, agent \( n \) wants to maximize her expected utility of taking action \( a_n \) given as follows.

\[
\max_{a_n \in \{0, 1\}} \left[ r_n u^1(a_n | s_n, I(n)) + (1 - r_n) u^0(a_n | s_n, I(n)) \right].
\]

A strategy profile is a sequence of strategies \( \sigma = \{\sigma_n\}_{n \in \mathbb{N}} \) where \( \sigma_n : \mathbb{I}(n) \times [0, 1] \rightarrow \{0, 1\} \). Given a strategy profile \( \sigma \), since the realization of information sets is stochastic, the decision sequence \( \{a_n\}_{n \in \mathbb{N}} \) is a stochastic process, and the measure generated on the sequence of actions in \( \{0, 1\}^\infty \) is \( \mathbb{P}_\sigma^\infty \).
**Definition 3.2.** A strategy profile $\sigma^*$ is a pure-strategy Perfect Bayesian Equilibrium of this game of social learning if for each $n \in \mathbb{N}$, $\sigma^*_n$ maximizes the expected payoff as in (2) given the strategies of other agents $\sigma^*_{-n}$ preceding her.

In the rest of the paper, equilibrium will mean Perfect Bayesian Equilibrium (PBE) strategy. Clearly, in this case an equilibrium strategy exists and the set of equilibria of the game is non-empty. Given the sequence of strategies $\{\sigma^*_1, \ldots, \sigma^*_{n-1}\}$, the maximization problem has a solution for each $n \in \mathbb{N}$ and $I(n) \in \mathbb{I}(n)$ because the set of possible actions is finite. Proceeding inductively and choosing one of the two possible actions in case of indifference determines an equilibrium. Hence we have the following proposition.

**Proposition 3.1.** There exists a pure-strategy Perfect Bayesian Equilibrium.

My main focus is how this equilibrium strategy influences information aggregation and learning about the underlying state of the world. Let $\sigma^*(\theta)$ be the PBE strategy if the state $\theta$ were known. Then, I define social learning (or asymptotic learning) as follows.

**Definition 3.3.** Given the signal structure and the network structure, asymptotic learning occurs under equilibrium strategy $\sigma^*$ if

$$\lim_{n \to \infty} Pr(a^*_n = a^*_n(\theta)) = 1,$$

where $a^*_n$ and $a^*_n(\theta)$ are the decisions corresponding to the strategy profiles $\sigma^*_n$ and $\sigma^*_n(\theta)$, respectively for a particular realization of information set $I(n)$.

So learning means that the probability of taking the correct decision converges to 1.

### 4 Strong Externality with Symmetric Payoffs (SES)

To understand the implications of case SES, as in Section 3.1 on social learning, first I have to define the network structure. I consider two possible forms.

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3 More general network structures are considered in section 4.8.
Social Learning with Payoff Externality

N1 \( \Lambda(n) = \{n - 1\} \quad \forall n > 1 \) and \( \Lambda(1) = \Phi \).

N2 \( \Lambda(n) = \{n - 2, n - 1\} \quad \forall n > 2 \), \( \Lambda(2) = \{1\} \) and \( \Lambda(1) = \Phi \).

The first network structure, N1, is one where, out of the entire history of actions taken by all preceding agents, the only thing an individual observes is the action of the agent just preceding her. This means that the first agent in this sequence is a special case, as there is no action being taken before that. So the first agent observes just her own action.

The second network structure, N2, represents better observability on the part of the agents as each one observe the actions of two players preceding her. The first and second agents in this case are special: Whereas, the first agent only observes her own action, the second action is basically reduced to that of type N1.

Next, I consider the simple version of the payoff function under SES. Such a payoff structure says that its better to do as your predecessor did if the state is 1 and do the opposite if the state is 0.

4.1 Learning under N1

Given the payoff and the network structure N1, its easy to characterize the individual decisions in equilibrium. Because of the network structure, there will be two forms of neighborhood belief. Either, an agent \( n \) observes the preceding agent taking action 0 or she observes her taking action 1. And accordingly, she gets her information from her neighborhood. Let

\[
q(n, 0) = P(\theta = 1 | a_{n-1} = 0), \quad q(n, 1) = P(\theta = 1 | a_{n-1} = 1).
\]

Given this, it is easy to see that the equilibrium strategy is a cut-off strategy\(^4\) where the cut-offs are decided by the neighborhood beliefs given above. Note that there will be two different cut-offs depending on the action of the preceding agent. The following lemma

\(^4\)A cutoff strategy means that there exists a \( p^* \in [0, 1] \) such that if private belief of an agent \( n \) is \( p_n \leq p^* \), she will take action \( a \in \{0, 1\} \) and if \( p_n > p^* \), she takes action \((1 - a)\).
Lemma 4.1. Let $\sigma^*$ be a PBE of the game. With network structure $N$, let $I(n) \in I(n)$ be an information set of agent $n$. Then, the action of the agent $n$, $a^*_n = \sigma^*(I(n))$, is as follows.

$$a^*_n = \begin{cases} a_{n-1} & \text{if } p_n > 1 - q(n,a_{n-1}) \\ 1 - a_{n-1} & \text{if } p_n \leq 1 - q(n,a_{n-1}). \end{cases}$$

Proof. In Appendix B.

Once the optimum action of agents have been characterized in terms of realization of the private belief $p_n$, it is now possible to see how the probability of taking the correct decision $Pr(a^*_n = a^*_n(\theta))$ is determined. For this I need to consider two cases.

**Case 1:** ($q(n,1) > \frac{1}{2}$) The optimum action for agent $n$ can be described in the following form.

$$a^*_n = \begin{cases} a_{n-1} & \text{if } p_n > 1 - q(n,0) \\ 1 & \text{if } p_n \in (1 - q(n,1),1 - q(n,0)] \\ 1 - a_{n-1} & \text{if } p_n \leq 1 - q(n,1). \end{cases}$$

This means the probability of taking correct decision can be broken down into two parts conditional on whether $\theta = 0$ or $\theta = 1$.

$$Pr(a^*_n = a^*_n(\theta)) = Pr(\theta = 0)\{Pr(p_n \leq 1 - q(n,1)|\theta = 0)$$

$$+ Pr(p_n \in (1 - q(n,1),1 - q(n,0)],a_{n-1} = 0|\theta = 0)\}$$

$$+ Pr(\theta = 1)\{Pr(p_n > 1 - q(n,0)|\theta = 1)$$

$$+ Pr(p_n \in (1 - q(n,1),1 - q(n,0]),a_{n-1} = 1|\theta = 1)\}$$

Since private signals are independently distributed, the private belief $p_n$ of agent $n$ is also independently distributed from actions taken by previous agents. So we have the
following.

\[
Pr(a_n^* = a_n^*(\theta)) = \frac{1}{2} G_0(1 - q(n, 1)) + \frac{1}{2} [1 - G_1(1 - q(n, 0))]
\]

\[
+ \frac{1}{2} Pr(a_{n-1} = 0|\theta = 0) [G_0(1 - q(n, 0)) - G_0(1 - q(n, 1))]
\]

\[
+ \frac{1}{2} Pr(a_{n-1} = 1|\theta = 1) [G_1(1 - q(n, 0)) - G_1(1 - q(n, 1))].
\]

Next, the neighborhood beliefs are symmetric in the sense that

\[
q(n, 1) = Pr(\theta = 1|a_{n-1} = 1)
\]

\[
= Pr(a_{n-1} = 1|\theta = 1) \quad \text{[since } Pr(a_{n-1} = 1) = Pr(\theta = 1) = \frac{1}{2} \text{]}
\]

\[
= 1 - Pr(a_{n-1} = 0|\theta = 1)
\]

\[
= 1 - Pr(\theta = 1|a_{n-1} = 0)
\]

\[
= 1 - q(n, 0).
\]

Further in both cases \(q(1, \cdot) = \frac{1}{2}\). So using private belief distributions as in (1) we finally have

\[
Pr(a_n^* = a_n^*(\theta)) = 1 - q(n, 1) + q(n, 1)^2.
\]

**Case 2:** \((q(n, 1) \leq \frac{1}{2})\) The optimum action for agent \(n\) can be described in the following form.

\[
a_n^* = \begin{cases} 
  a_{n-1} & \text{if } p_n > 1 - q(n, 1) \\
  0 & \text{if } p_n \in (1 - q(n, 0), 1 - q(n, 1)] \\
  1 - a_{n-1} & \text{if } p_n \leq 1 - q(n, 0).
\end{cases}
\]

Following a similar line as in case 1, we have

\[
Pr(a_n^* = a_n^*(\theta)) = 1 - q(n, 0) + q(n, 0)^2
\]

\[
= 1 - q(n, 1) + q(n, 1)^2.
\]

Hence the following lemma.

**Lemma 4.2.** Let \(\sigma^*\) be a PBE of the game with network structure \(N_1\). Let \(I(n)\) be the information set of agent \(n\), \((G_0, G_1)\) the private belief distribution and the action of the agent \(n\) is \(a_n^* = \sigma^*(I(n))\). Then

\[
Pr(a_n^* = a_n^*(\theta)) = 1 - q(n, 1) + q(n, 1)^2.
\]
4.2 Benchmark

To understand the impact of externality on learning, I introduce the benchmark case, NES with the same network structure N1. From now, any variable from the benchmark case will be represented with superscript B (eg. \( q^B(n, .) \) is the neighborhood belief of agent n from the benchmark model).

The equivalent of Lemma (4.1) in the benchmark case is as follows.

If \( a_{n-1} = 0 \), then

\[
\begin{align*}
a^*_n &= \begin{cases} 
0 & \text{if } \frac{p_n}{1 - q(n, 0)} \leq 1 \ \text{or } \frac{p_n}{1 - q(n, 0)} > 1 \end{cases} \\
&= \begin{cases} 
1 & \text{if } \frac{p_n}{1 - q(n, 0)} \leq 1 \ \text{or } \frac{p_n}{1 - q(n, 0)} > 1 
\end{cases}
\end{align*}
\]

If \( a_{n-1} = 1 \), then

\[
\begin{align*}
a^*_n &= \begin{cases} 
1 & \text{if } \frac{p_n}{1 - q(n, 1)} \leq 1 \ \text{or } \frac{p_n}{1 - q(n, 1)} > 1 \\
0 & \text{if } \frac{p_n}{1 - q(n, 1)} \leq 1 \ \text{or } \frac{p_n}{1 - q(n, 1)} > 1 
\end{cases}
\end{align*}
\]

Note that the result in Lemma (4.2) will also hold in the benchmark case.

4.3 Benchmark and SES: Comparison under N1

**Proposition 4.1.** Under the network structure N1, private belief distributions as in (1):

(a) There is social learning in the benchmark case.

(b) In the case with externality of the type SES, there is no social learning, i.e.

\[
\lim_{n \to \infty} \Pr(a^*_n = a^*_n(\theta)) < 1.
\]

**Proof.** In Appendix B. \qed

Applying Lemma (4.2), the probability of taking the correct decision in case with externality evolves as follows.

\[
\Pr(a^*_n = a^*_n(\theta)) = 1 - q(n, 1) + q(n, 1)^2
\]
and for the benchmark case,

\[ Pr^B(a_n^* = a_n^*(\theta)) = 1 - q^B(n, 1) + q^B(n, 1)^2. \]

And for \( n > 2 \) these cutoffs take the following form.

\[
q(n, 1) = Pr(\theta = 1|a_{n-1}^* = 1, a_{n-2}^* = 0)Pr(a_{n-2}^* = 0|a_{n-1}^* = 1) \\
+ Pr(\theta = 1|a_{n-1}^* = 1, a_{n-2}^* = 1)Pr(a_{n-2}^* = 1|a_{n-1}^* = 1).
\]

Using the distribution functions for private belief, we have

\[
q(n, 1) = (1 - q(n-1, 1)^2(3 - 2q(n-1, 1))) \frac{1 - q(n-1, 1)}{3 - 2q(n-1, 1)} \\
+ q(n-1, 1)^2(3 - 2q(n-1, 1)) \frac{2 - q(n-1, 1)}{3 - 2q(n-1, 1)},
\]

and

\[
q^B(n, 1) = (1 - q^B(n-1, 1)^2(3 - 2q^B(n-1, 1))) \frac{1 + q^B(n-1, 1)}{1 + 2q^B(n-1, 1)} \\
+ q^B(n-1, 1)^2(3 - 2q^B(n-1, 1)) \frac{2 - q^B(n-1, 1)}{3 - 2q^B(n-1, 1)}.
\]

So social learning occurs when the cutoffs \( q(n, 1) \) and \( q^B(n, 1) \) converge to 1. These cutoffs are, by definition, iterative. So, if is the difference \( q(n, 1) - q(n-1, 1) \) is positive, this will ensure that, given the upper bound of 1, the cutoffs will converge to 1 and so will the probability of taking the correct decision.

\[ \Delta_n^B \]

Figure 3: Change in \( q^B \)

---

\(^5\)A more detailed derivation can be found in Appendix B
As can be seen in figure (3), this is in fact the case for the benchmark payoff. However, as in figure (4), in the presence of externality, the difference $q(n, 1) - q(n - 1, 1)$ is in fact negative for $q(n - 1, 1)$ greater than $1/2$ and negative for $q(n - 1, 1)$ less than $1/2$ which means that $q(n, 1)$ never improves from $1/2$.

And this is exactly what can be seen in figure (5). And hence the above proposition.
4.4 Learning under N2

Next, I consider the learning process when the network structure is N2. Now the neighborhood belief can take the following forms.

\[ q(n, S) = P(\theta = 1 | a_{n-1} \neq a_{n-2}) \]
\[ q(n, NS) = P(\theta = 1 | a_{n-1} = a_{n-2}) \]

where \( S \) represents switching or changing actions between \( a_{n-1} \) and \( a_{n-2} \) and \( NS \) is the case of no switch or no change in actions \( a_{n-1} \) and \( a_{n-2} \). The equivalent of Lemma (4.2) in this case is as follows.

\[ Pr(a_n^* = a_n^*(\theta)) = 1 - q(n, NS) + q(n, NS)^2. \] (5)

**Proposition 4.2.** Under the network structure N2, private belief distributions as in (1), there is social learning even with externality of type SES.

**Proof.** In Appendix B. \( \square \)

In this case, the probability of taking the correct decision depends on the cutoff \( q(n, NS) \). As in proposition (4.1), it is, by definition, iterative and has an upper bound of 1. I show that similar to figure (3), the difference \( q(n, NS) - q(n-1, NS) \) is positive which ensures that \( q(n, NS) \) converges to 1 and so does the probability of taking the correct decision.

4.4.1 Bounded Private Beliefs

The learning process, that we had in the last section, will be disturbed if the private beliefs are bounded. To understand this, let me consider a generalized distribution of private beliefs \((G_0, G_1)\) but with support \([\underline{\beta}, \overline{\beta}]\) where \(0 < \underline{\beta} < \overline{\beta} < 1\).

**Proposition 4.3.** Under network structure N2 and bounded private belief distributions \((G_0, G_1)\) with support \([\underline{\beta}, \overline{\beta}]\) \((0 < \underline{\beta} < \overline{\beta} < 1)\), with externality type SES, there is no social learning.
With bounded private beliefs, I show that once the cutoff $q(n,NS)$ reaches the bound of $1 - \beta$, no further improvement of the cutoff is possible. Hence, the probability of taking the correct decision can never converge to 1.

### 4.5 Speed of Learning

Proposition (4.2) shows that with network structure N2 and unbounded private beliefs, the probability of taking the correct decision converges to 1. But what is the rate at which this convergence takes place? To find this rate I first define the rate of convergence of a sequence and then use that to find the speed of learning.

**Definition 4.1.** Suppose a sequence $\{x_k\}$ converges to the number $L$.

1. We say that this sequence converges with order $z$ to $L$, if there exists a number $\mu \in (0, 1)$ such that

   $$\lim_{k \to \infty} \frac{|x_{k+1} - L|}{|x_k - L|^z} = \mu$$

   The number $\mu$ is the rate of convergence.

2. If $\mu = 1$, $z = 1$ and

   $$\lim_{k \to \infty} \frac{|x_{k+2} - x_{k+1}|}{|x_{k+1} - x_k|} = 1$$

   then it is said that the sequence converges logarithmically to $L$.

With this definition and the results in proposition (4.2), I can now find the speed of learning.

**Proposition 4.4.** Under the network structure N2, private belief distributions as in (1), social learning proceeds logarithmically. That is, the probability of taking the correct decision converges logarithmically to 1.

**Proof.** In Appendix B. □
4.6 Stochastic Network Structure

As a direct consequence of results in sections 4.3 and 4.4, the question that immediately comes to mind is what happens when the network structure is neither purely $N_1$ nor $N_2$ but for each agent the neighborhood is a random draw from those two types of neighborhoods. The next proposition characterizes the nature of such a random draw that will lead to social learning.

**Proposition 4.5.** Let $\varepsilon_n$ be the probability that agent $n$ has her neighborhood according to network structure $N_1$, it being according to $N_2$ otherwise. Private belief distributions are as in (1). Let there be $\varepsilon^* \geq 0$ such that

$$\lim_{n \to \infty} \varepsilon_n = \varepsilon^*.$$  

(6)

Then there will be social learning under externality of type SES if and only if $\varepsilon^* = 0$.

**Proof.** In Appendix B.

If an agent has a network structure of type N1, her cutoff $q(n,1)$ will follow the same pattern as in proposition (4.1). That is, these agents fail to capture any information from the preceding agents. This means that when an agent of network structure N2 wants to update her belief by observing the action of agents in her neighborhood, she has to discount the fact that there is a non-zero probability of these agents being of type N1 and hence their actions not capturing any information from agents before them. This makes the cutoff $q(n,NS)$ always bounded away from 1 and hence the probability of taking the correct decision also remains bounded away from 1.

But even when the nature of the random draw of neighborhoods is such that social learning is not possible, we can have some idea about how close to social learning can these agents get. The next proposition deals with that particular question.

**Proposition 4.6.** Let $\varepsilon_n$ be the probability that agent $n$ has her neighborhood according to network structure $N_1$, it being according to $N_2$ otherwise. Private belief distributions
are as in (1). Let there be \( \varepsilon^* \geq 0 \) such that
\[
\lim_{n \to \infty} \varepsilon_n = \varepsilon^*.
\]
(7)
Then for any \( 0 \leq \delta < \frac{1}{8} \), there is an \( \bar{\varepsilon} \), \( 0 \leq \bar{\varepsilon} \leq 1 \) such that
\[
\varepsilon^* \geq \bar{\varepsilon} \implies \lim_{n \to \infty} \Pr(a_n^* = a_n^*(\theta)) \leq 1 - \delta.
\]

Proof. In Appendix B.

But what happens, with random observability, even though an agent doesn’t know the actions of all agents preceding her, she knows what type of observability they had, N1 or N2? The next proposition shows that in such a case, we will have social learning for agents with observability type N2, even though agents with observability type N1 can’t add any information to the existing belief. This happens due to the fact that when the type of observability of all preceding agents is known, the belief cutoffs become deterministic.

**Proposition 4.7.** Let \( \varepsilon \) be the probability that agent \( n \) has her neighborhood according to network structure N1, it being according to N2 otherwise. Private belief distributions are as in (1). In addition, every agent knows the type of network all preceding agents have. Then there will be social learning under externality of type SES but only for agents with network structure N2.

Proof. In Appendix B.

As was the case in proposition (4.5), even when the type of network of all preceding agents is known, an agent with network structure N1 fails to accumulate any information from preceding agents and hence this subset of agents doesn’t experience social learning. However, the situation of the agents of network type N2 is different. Since, the network type of all previous agents is known, for this subset of agents, the cutoff \( q(n, NS) \) is deterministic and so no information is lost between two N2 type agents. And when there are two consecutive N2 type agents, we see an improvement on that information. This process ensures that this subset of agents experiences social learning.
4.7 Example

As in proposition[4.5], suppose we have a sequence of agents taking an action from a binary choice \( \{0, 1\} \), which is based on the belief about the underlying state and the action taken by previous agents. The payoff structure is according to externality of type SES. The observability of an agent is randomly determined. With probability \( \varepsilon = 0.1 \), her observability is of type N1, and with probability \( (1 - \varepsilon) = 0.9 \), her observability is of type N2. The two exceptions to this random draw of observability for each agent are the first two agents in the sequence.

Now suppose the underlying state is \( \theta = 1 \). So given the framework and a random draw of private beliefs for each agent according to (1), we will be interested in knowing how the neighborhood belief and the posterior belief behave over time and how does that get
reflected in the clustering of actions by the agents. One thing to keep in mind is that given the state $\theta = 1$, the correct decision will be either everyone clustering in action 1 or everyone taking action 0.

![Figure 7: Evolution of beliefs and actions with non-random observability](image)

(a) Posterior belief  
(b) Actions  
(c) Neighborhood belief

The result of such a randomly generated sequence of private beliefs is shown in figure (6). As can be seen in (c), because there is a positive probability of the arrival of an agent with observability of type N1 (however small that probability be), the neighborhood belief cutoffs can never converge to 0 or 1. The result is that not only can the agents of type N1 make mistakes in their action, but even the well informed agents of type N2 can make mistakes and end up choosing the wrong action (which is the action of not doing what your predecessor did). And thus we see so many change in actions between 0 and 1 between successive agents.
On the other hand, when \( \varepsilon = 0 \), even if initially, actions might change between successive agents, over time, as the neighborhood belief cutoffs converge along with the posterior belief, it becomes more and more likely that the actions of successive agents converges to 0 or 1.

### 4.8 Towards More Generalization

An immediate possible extension to proposition(4.5) is what happens when people observe something more than \( N_2 \). To understand this, I first consider a more generalized network structure.

\[ N_k \text{, } \Lambda(n) = \{n - k, \ldots, n - 1\} \forall n > k, \Lambda(n) = \{1, \ldots, k - 1\} \forall 1 < n \leq k \text{ and } \Lambda(1) = \emptyset. \]

**Lemma 4.3.** For any \( n \in \mathbb{N} \), any network structure \( N_k \), (where \( k \in \mathbb{N} \) and \( k > 1 \)) and any equilibrium \( \sigma^* \)

\[
Pr(a_n^* = a_n^*(\theta)|N_k) \geq Pr(a_n^* = a_n^*(\theta)|N_2).
\]

**Proof.** In Appendix B. \( \square \)

This result follows from the fact that the probability of taking the correct decision is in fact the solution to a maximization problem which means that given a bigger information set, the resulting probability will be weakly greater than the one with the smaller information set.

Three immediate results follow from this lemma.

**Corollary 4.1.** Under the network structure \( N_k \forall k > 1 \), private belief distributions as in (1), there is social learning with externality of type SES.

**Corollary 4.2.** Let \( \varepsilon_n \) be the probability that agent \( n \) has her neighborhood according to network structure \( N_1 \), it being according to \( N_k \) (\( \forall k > 1 \)) otherwise. Private belief
distributions are as in (1). Let there be \( \varepsilon^* \geq 0 \) such that
\[
\lim_{n \to \infty} \varepsilon_n = \varepsilon^*.
\]
Then there will be social learning with externality of type SES if and only if \( \varepsilon^* = 0 \).

**Corollary 4.3.** Let \( \varepsilon \) be the probability that agent \( n \) has her neighborhood according to network structure \( N_1 \), it being according to \( N_k \) \( (\forall k > 1) \) otherwise. Private belief distributions are as in (1). In addition, every agent knows the type of network all preceding agents have. Then there will be social learning under externality of type SES but only for agents with network structure \( N_k \).

### 5 Conclusion

In this paper, I studied the problem of Bayesian learning and information accumulation over a general social network in the presence of payoff externality from the actions of previous agents. A large literature has studied Bayesian learning in sequential-move games. However, most of this literature has considered perfect observability of the agents. Further the actions of agents do not, in any way, affect the payoff of agents following them. In this category falls classical works like Banerjee (1992), Bikhchandani et al. (1992), and Smith and Sørensen (2000). Some like Acemoglu et al. (2011) have considered limited observability of the agents, but they still ignore the possibility of payoff externality.

In many relevant situations, not only do the agents have limited observability about the history of actions taken in the past, their own payoff is also affected (either directly or indirectly) by those past actions. This is what I set out to study in this paper, and understand the implications on social learning. I showed that in the absence of externality, even with limited observability, there will be social learning. However, introduction of payoff externality creates problems. I was able to describe the network structures under which social learning is possible and when it isn’t. These results were derived using a symmetric private belief distribution. So future endeavors should involve understanding the impact of a more general private belief distribution.
Further, this kind of externality and information flow can also be studied in the setting of a static network of individuals taking repeated actions over a long duration. And the idiosyncrasies of such network structures can give further insight on the effect of different forms of externality on information flow and accumulation.

There can also be implications on the empirical side which can be extensions of the current results. An obvious testable result, which follows from my discussions in the introduction, is the sale of video games. It is to be seen, for the same quality, how much the sale volumes differ for exclusively single player versus multi-player games.

Appendix A

Let $\mathbb{I}(n)$ be the set of all possible $I(n)$ allowed under a network structure. Then $\forall n$, there is a payoff-relevant partition of $\mathbb{I}(n)$ given by $\{\mathbb{I}_0(n), \mathbb{I}_1(n)\}$. Finally, the payoff of agent $n$ is as follows.

If $I(n) \in \mathbb{I}_i(n), i \in \{0, 1\}$,

$$u^0(a_n|I(n)) = \begin{cases} U^0_{i0} & a_n = 0 \\ U^0_{i1} & a_n = 1, \end{cases}$$

(8)

$$u^1(a_n|I(n)) = \begin{cases} U^1_{i0} & a_n = 0 \\ U^1_{i1} & a_n = 1. \end{cases}$$

(9)

where $U^\theta_{ij}$ is the payoff of taking action $j \in \{0, 1\}$ in state $\theta$ when the observed information is in partition $i \in \{0, 1\}$. Given this payoff structure, there are four non-trivial situations that I consider.

**NES** $U^\theta_{ki} > U^\theta_{kj} = 0, U^\theta_{0i} = U^\theta_{1i}, \forall i, j, k \in \{0, 1\}, i \neq j, i = \theta, \theta \in \Theta.$

This is the case of no externality with symmetric payoffs and the corresponding game is a generalized version of the social learning game in [Acemoglu et al., (2011)].

The particular learning dynamics in this case depends on the network structure and under certain conditions information aggregation is possible. A simplified version
of this payoff structure would be as follows.

\[
\begin{align*}
  u^0(a_n) &= \begin{cases} 
    1 & a_n = 0 \\
    0 & a_n = 1,
  \end{cases} & u^1(a_n) &= \begin{cases} 
    0 & a_n = 0 \\
    1 & a_n = 1.
  \end{cases}
\end{align*}
\]

**WEA** \( U_{ki}^\theta > U_{kj}^\theta = 0, \ U^\theta_{0i} \neq U^\theta_{1i}, \forall i, j, k \in \{0,1\}, \ i \neq j, \ i = \theta, \ \theta \in \Theta. \)

This is the case of weak externality with asymmetric payoffs. A simplified version of this payoff structure would be as follows.

If \( a_{n-1} = 0 \), then

\[
\begin{align*}
  u^0(a_n) &= \begin{cases} 
    1 & a_n = 0 \\
    0 & a_n = 1,
  \end{cases} & u^1(a_n) &= \begin{cases} 
    0 & a_n = 0 \\
    1 & a_n = 1.
  \end{cases}
\end{align*}
\]

Otherwise if \( a_{n-1} = 1 \), then

\[
\begin{align*}
  u^0(a_n) &= \begin{cases} 
    1 & a_n = 0 \\
    0 & a_n = 1,
  \end{cases} & u^1(a_n) &= \begin{cases} 
    0 & a_n = 0 \\
    1 + \gamma & a_n = 1.
  \end{cases}
\end{align*}
\]

where \( \gamma > 0 \) and \( a_0 = 1 \).

**SES** \( U^\theta_{1i} > U^\theta_{1j}, \ U^\theta_{0i} < U^\theta_{0j}; \ U^\theta_{1i} = U^\theta_{0i} = 0, \ U^\theta_{0j} = U^\theta_{1j}, \forall i, j, k \in \{0,1\}, \ i \neq j, \ i = \theta, \ \theta \in \Theta. \)

This represents strong externality with symmetric payoff. The example of farmland and decision of choosing a crop to plant fits into this category. A simplified version of this payoff structure is as follows.

If \( a_{n-1} = 0 \), then

\[
\begin{align*}
  u^0(a_n) &= \begin{cases} 
    0 & a_n = 0 \\
    1 & a_n = 1,
  \end{cases} & u^1(a_n) &= \begin{cases} 
    1 & a_n = 0 \\
    0 & a_n = 1.
  \end{cases}
\end{align*}
\]

Otherwise if \( a_{n-1} = 1 \), then
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\[ u^0(a_n) = \begin{cases} 
1 & a_n = 0 \\
0 & a_n = 1, 
\end{cases} \quad u^1(a_n) = \begin{cases} 
0 & a_n = 0 \\
1 & a_n = 1. 
\end{cases} \]

where \( a_0 = 1. \)

**SEA** \( U^\theta_{ij} > U^\theta_{1j}, \ U^\theta_{1i} < U^\theta_{0i}; \ U^\theta_{1j} = U^\theta_{0j} = 0, \ U^\theta_{1j} \neq U^\theta_{0j}; \ \forall i, j, k \in \{0, 1\}, \ i \neq j, \ i = \theta, \ \theta \in \Theta. \)

This represents strong externality with asymmetric payoff. A simplified version of this payoff structure is as follows.

If \( a_{n-1} = 0, \) then

\[ u^0(a_n) = \begin{cases} 
0 & a_n = 0 \\
1 & a_n = 1, 
\end{cases} \quad u^1(a_n) = \begin{cases} 
1 & a_n = 0 \\
0 & a_n = 1. 
\end{cases} \]

Otherwise if \( a_{n-1} = 1, \) then

\[ u^0(a_n) = \begin{cases} 
1 & a_n = 0 \\
0 & a_n = 1, 
\end{cases} \quad u^1(a_n) = \begin{cases} 
0 & a_n = 0 \\
1 + \gamma & a_n = 1. 
\end{cases} \]

where \( \gamma > 0 \) and \( a_0 = 1. \)

**Appendix B**

**Proof of Lemma 4.1.** Let \( a_{n-1} = 1. \) Then to maximize expected payoff as in (2),

\[ a_n^* = \begin{cases} 
0 & \text{if } 0 \leq r_n \leq \frac{1}{2} \\
1 & \text{if } \frac{1}{2} < r_n \leq 1. 
\end{cases} \]

Now with \( a_{n-1} = 1, \) \( q_n = q(n, 1) \) and so

\[ r_n \leq \frac{1}{2} \implies p_n \leq 1 - q(n, 1). \]

Similarly,

\[ r_n > \frac{1}{2} \implies p_n > 1 - q(n, 1). \]
The case with $a_{n-1} = 0$ is analogous except that now $q_n = q(n,0)$.

**Proof of Proposition 4.1.** The neighborhood beliefs for $n > 2$ in the two cases evolve as follows.

For $q(n,1) = Pr(\theta = 1 | a_{n-1}^* = 1)$

$$q(n,1) = Pr(\theta = 1 | a_{n-1}^* = 1, a_{n-2}^* = 0)Pr(a_{n-2}^* = 0 | a_{n-1}^* = 1)$$

$$+ Pr(\theta = 1 | a_{n-1}^* = 1, a_{n-2}^* = 1)Pr(a_{n-2}^* = 1 | a_{n-1}^* = 1)$$

$$= Pr(a_{n-2}^* = 0 | a_{n-1}^* = 1) \left(1 + \frac{G_0(1-q(n-1,0))}{G_1(1-q(n-1,0))} \frac{1-q(n-1,0)}{q(n-1,0)}\right)^{-1}$$

$$+ Pr(a_{n-2}^* = 1 | a_{n-1}^* = 1) \left(1 + \frac{1-G_0(1-q(n-1,1))}{1-G_1(1-q(n-1,1))} \frac{1-q(n-1,1)}{q(n-1,1)}\right)^{-1},$$

and

$$q^B(n,1) = Pr(a_{n-2}^* = 0 | a_{n-1}^* = 1) \left(1 + \frac{1-G_0(1-q(n-1,0))}{1-G_1(1-q(n-1,0))} \frac{1-q(n-1,0)}{q(n-1,0)}\right)^{-1}$$

$$+ Pr(a_{n-2}^* = 1 | a_{n-1}^* = 1) \left(1 + \frac{1-G_0(1-q(n-1,1))}{1-G_1(1-q(n-1,1))} \frac{1-q(n-1,1)}{q(n-1,1)}\right)^{-1}.$$

Further, in both cases, $Pr(a_n^* = 0) = Pr(a_n^* = 1) = \frac{1}{2}$, for all $n$. So,

$$Pr(a_{n-2}^* = 1 | a_{n-1}^* = 1) = Pr(a_{n-1}^* = 1 | a_{n-2}^* = 1)$$

$$= (1-q(n-1,1))[1-G_0(1-q(n-1,1))]$$

$$+ q(n-1,1)[1-G_1(1-q(n-1,1))]$$

$$= q(n-1,1)^2(3-2q(n-1,1)),$$

and for case B,

$$Pr(a_{n-2}^* = 1 | a_{n-1}^* = 1) = Pr(a_{n-1}^* = 1 | a_{n-2}^* = 1)$$

$$= (1-q^B(n-1,1))[1-G_0(1-q^B(n-1,1))]$$

$$+ q^B(n-1,1)[1-G_1(1-q^B(n-1,1))]$$

$$= q^B(n-1,1)^2(3-2q^B(n-1,1)).$$
Using these I have, for \( n > 2 \)

\[
q(n, 1) = (1 - q(n - 1, 1)^2(3 - 2q(n - 1, 1))) \frac{1 - q(n - 1, 1)}{3 - 2q(n - 1, 1)} + q(n - 1, 1)^2(3 - 2q(n - 1, 1)) \frac{2 - q(n - 1, 1)}{3 - 2q(n - 1, 1)},
\]

and

\[
q^B(n, 1) = (1 - q^B(n - 1, 1)^2(3 - 2q^B(n - 1, 1))) \frac{1 + q^B(n - 1, 1)}{1 + 2q^B(n - 1, 1)} + q^B(n - 1, 1)^2(3 - 2q^B(n - 1, 1)) \frac{2 - q^B(n - 1, 1)}{3 - 2q^B(n - 1, 1)}.
\]

Also, applying Lemma 4.2, the probability of taking the correct decision in case with externality evolves as follows.

\[
Pr(a_n^* = a_n^*(\theta)) = 1 - q(n, 1) + q(n, 1)^2,
\]

and for case B,

\[
Pr^B(a_n^* = a_n^*(\theta)) = 1 - q^B(n, 1) + q^B(n, 1)^2.
\]

Next, I show that \( q^B(n, 1) \) is strictly increasing for all \( q^B(n - 1, 1) \in (0, 1) \) and \( q(n, 1) \) is not.

\[
q^B(n, 1) - q^B(n - 1, 1) = (1 - q^B(n - 1, 1)^2(3 - 2q^B(n - 1, 1))) \frac{1 + q^B(n - 1, 1)}{1 + 2q^B(n - 1, 1)} + q^B(n - 1, 1)^2(3 - 2q^B(n - 1, 1)) \frac{2 - q^B(n - 1, 1)}{3 - 2q^B(n - 1, 1)} - q^B(n - 1, 1)
\]

\[
= \frac{1 - 3q^B(n - 1, 1)^2 + 2q^B(n - 1, 1)^3}{1 + 2q^B(n - 1, 1)}
\]

\[
= \frac{1 - q^B(n - 1, 1)^2(3 - 2q^B(n - 1, 1))}{1 + 2q^B(n - 1, 1)} > 0 \forall q^B(n - 1, 1) \in (0, 1).
\]

And

\[
q(n, 1) - q(n - 1, 1) = (1 - q(n - 1, 1)^2(3 - 2q(n - 1, 1))) \frac{1 - q(n - 1, 1)}{3 - 2q(n - 1, 1)} + q(n - 1, 1)^2(3 - 2q(n - 1, 1)) \frac{2 - q(n - 1, 1)}{3 - 2q(n - 1, 1)} - q(n - 1, 1)
\]

\[
= \frac{1 - 4q(n - 1, 1) + 5q(n - 1, 1)^2 - 2q(n - 1, 1)^3}{3 - 2q(n - 1, 1)}
\]

\[
= \frac{(1 - q(n - 1, 1))(1 - q(n - 1, 1))(3 - 2q(n - 1, 1))}{3 - 2q(n - 1, 1)}.
\]
So

\[
q(n, 1) - q(n - 1, 1) = \begin{cases} 
> 0 & \text{if } q(n - 1, 1) \in (0, 0.5) \\
= 0 & \text{if } q(n - 1, 1) = 0.5 \\
< 0 & \text{if } q(n - 1, 1) \in (0.5, 1).
\end{cases}
\]

Since \(q^*(n, 1)\) is strictly increasing and has an upper bound, it converges to its upper bound 1. And hence probability of taking the correct decision also converges to 1. But \(q(n, 1)\) always remains 0.5 and hence the probability of taking the correct decision in this case never converges to 1. These changes are shown in Figure 5. And so, compared to case B, no learning occurs in case A.

\[\square\]

**Proof of Proposition 4.2.** The neighborhood belief evolves as follows

\[
q(n, NS) = Pr(a_{n-3}^* = a_{n-2}^* | a_{n-2}^* = a_{n-1}^*) \left( 1 + \frac{1 - G_0(1 - q(n - 1, NS))}{1 - G_1(1 - q(n - 1, NS))} \frac{1 - q(n - 1, NS)}{q(n - 1, NS)} \right)^{-1}
\]

\[+ Pr(a_{n-3}^* \neq a_{n-2}^* | a_{n-2}^* = a_{n-1}^*) \left( 1 + \frac{1 - G_0(1 - q(n - 1, S))}{1 - G_1(1 - q(n - 1, S))} \frac{1 - q(n - 1, S)}{q(n - 1, S)} \right)^{-1},
\]

and

\[
q(n, S) = Pr(a_{n-3}^* \neq a_{n-2}^* | a_{n-2}^* = a_{n-1}^*) \left( 1 + \frac{G_0(1 - q(n - 1, S))}{G_1(1 - q(n - 1, S))} \frac{1 - q(n - 1, S)}{q(n - 1, S)} \right)^{-1}
\]

\[+ Pr(a_{n-3}^* = a_{n-2}^* | a_{n-2}^* \neq a_{n-1}^*) \left( 1 + \frac{G_0(1 - q(n - 1, NS))}{G_1(1 - q(n - 1, NS))} \frac{1 - q(n - 1, NS)}{q(n - 1, NS)} \right)^{-1}.
\]

Now, \(\forall n\), because of symmetric payoffs, \(Pr(a_{n}^* = a_{n-1}^*) = Pr(a_{n}^* \neq a_{n-1}^*) = \frac{1}{2}\), so that \(q(n, S) = 1 - q(n, NS)\). Further,

\[
Pr(a_{n-3}^* = a_{n-2}^* | a_{n-2}^* = a_{n-1}^*) = Pr(a_{n-2}^* = a_{n-1}^* | a_{n-3}^* = a_{n-2}^*)
\]

\[= (1 - q(n - 1, NS))[1 - G_0(1 - q(n - 1, NS))] + q(n - 1, NS)[1 - G_1(1 - q(n - 1, NS))]
\]

\[= q(n - 1, NS)^2 (3 - 2q(n - 1, NS)).
\]

Using this in the neighborhood belief expression

\[
q(n, NS) = q(n - 1, NS)^2 (2 - q(n - 1, NS))
\]

\[+ (1 - q(n - 1, NS)^2 (3 - 2q(n - 1, NS))) \frac{1 + q(n - 1, NS)}{1 + 2q(n - 1, NS)}.
\]
The probability of taking the correct decision,

\[ \Pr(a_n^* = a_n^*(\theta)) = 1 - q(n,NS) + q(n,NS)^2. \]

Next,

\[
q(n,NS) - q(n-1,NS) = (1 - q(n-1,NS)^2)(3 - 2q(n-1,NS)) \frac{1 + q(n-1,NS)}{1 + 2q(n-1,NS)}
+ q(n-1,NS)^2(3 - 2q(n-1,NS)) \frac{2 - q(n-1,NS)}{3 - 2q(n-1,NS)}
- q(n-1,NS)
\]

\[
= \frac{1 - 3q(n-1,NS)^2 + 2q(n-1,NS)^3}{1 + 2q(n-1,NS)}
= \frac{1 - q(n-1,NS)^2(3 - 2q(n-1,NS))}{1 + 2q(n-1,NS)}
> 0 \forall q(n-1,NS) \in (0,1).
\]

So since, \(q(n,NS)\) is strictly increasing and has an upper bound, it converges to its upper bound 1. And hence the probability of taking the correct decision also converges to 1. So, with \(N2\), there is social learning. \(\Box\)

**Proof of Proposition 4.3.** As in proposition 4.2, the neighborhood belief evolves as follows.

\[
q(n,NS) = \Pr(a_{n-3}^* = a_{n-2}^*|a_{n-2}^* = a_{n-1}^*) \left( 1 + \frac{1 - G_0(1 - q(n-1,NS))}{1 - G_1(1 - q(n-1,NS))} \frac{1 - q(n-1,NS)}{q(n-1,NS)} \right)^{-1}
\]

\[
+ \Pr(a_{n-3}^* \neq a_{n-2}^*|a_{n-2}^* = a_{n-1}^*) \left( 1 + \frac{1 - G_0(1 - q(n-1,S))}{1 - G_1(1 - q(n-1,S))} \frac{1 - q(n-1,S)}{q(n-1,S)} \right)^{-1},
\]

with \(q(n,S) = 1 - q(n,NS)\) and

\[
\Pr(a_{n-3}^* = a_{n-2}^*|a_{n-2}^* = a_{n-1}^*) = \Pr(a_{n-2}^* = a_{n-1}^*|a_{n-3}^* = a_{n-2}^*)
= (1 - q(n-1,NS))[1 - G_0(1 - q(n-1,NS))]
+ q(n-1,NS)[1 - G_1(1 - q(n-1,NS))].
\]

Since \(G_0(\beta) = G_1(\beta) = 0\), for any \(1 > q(n-1,NS) \geq 1 - \beta\),

\[ q(n,NS) = q(n-1,NS) < 1. \]
Then, the probability of taking the correct decision,
\[ \Pr(a_n^* = a_n^*(\theta)) = \frac{1}{2} G_0(1 - q(n,NS)) + \frac{1}{2} [1 - G_1(1 - q(n,S))] \]
\[ + \frac{1}{2} \Pr(a_{n-1} \neq a_{n-2}|\theta = 0)[G_0(1 - q(n,S)) - G_0(1 - q(n,NS))] \]
\[ + \frac{1}{2} \Pr(a_{n-1} = a_{n-2}|\theta = 1)[G_1(1 - q(n,S)) - G_1(1 - q(n,NS))] \]
\[ \leq \frac{1}{2} [1 - G_1(1 - q(n,S))] + \frac{1}{2} q(n,NS)[G_0(1 - q(n,S)) + G_1(1 - q(n,S))] \]
\[ \leq \frac{1}{2} [1 + G_0(1 - q(n,S))] \leq 1. \]
So,
\[ \Pr(a_n^* = a_n^*(\theta)) < 1, \]
for all \( n \). The case for \( \bar{\beta} \) is analogous. Hence, there is no social learning with bounded private beliefs. 

**Proof of Proposition 4.4.** As in proposition 4.2, the probability of taking the correct decision is
\[ \Pr(a_n^* = a_n^*(\theta)) = 1 - q(n,NS) + q(n,NS)^2, \]
which converges to 1, where
\[ q(n,NS) = q(n - 1,NS)^2(2 - q(n - 1,NS)) \]
\[ + (1 - q(n - 1,NS)^2(3 - 2q(n - 1,NS))) \frac{1 + q(n - 1,NS)}{1 + 2q(n - 1,NS)}. \]
Now,
\[ \lim_{n \to \infty} \frac{1 - \Pr(a_{n+1}^* = a_{n+1}^*(\theta))}{1 - \Pr(a_n^* = a_n^*(\theta))} = \lim_{n \to \infty} \frac{q(n + 1,NS) - q(n + 1,NS)^2}{q(n,NS) - q(n,NS)^2} = 1. \]
And further,
\[ \lim_{n \to \infty} \frac{\Pr(a_{n+2}^* = a_{n+2}^*(\theta)) - \Pr(a_{n+1}^* = a_{n+1}^*(\theta))}{\Pr(a_{n+1}^* = a_{n+1}^*(\theta)) - \Pr(a_n^* = a_n^*(\theta))} = \lim_{n \to \infty} \frac{q(n + 1,NS) - q(n + 1,NS)^2 - q(n + 2,NS) + q(n + 2,NS)^2}{q(n,NS) - q(n,NS)^2 - q(n + 1,NS) + q(n + 1,NS)^2} = 1. \]
Hence, the probability of taking the correct decision converges logarithmically to 1. 

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Proof of Proposition (4.5). For any agent \( n \), her neighborhood can either be \( \Lambda_{N1}(n) = \{n - 1, n\} \) (with probability \( \epsilon_n \)) or \( \Lambda_{N2}(n) = \{n - 2, n - 1, n\} \) (with probability \( (1 - \epsilon_n) \)). This means that her neighborhood belief can either take the form

\[
q(n, S) = P(\theta = 1 | a_{n-1} \neq a_{n-2})
\]

and

\[
q(n, NS) = P(\theta = 1 | a_{n-1} = a_{n-2}),
\]

when her neighborhood is \( \Lambda_{N2}(n) \). However, when her neighborhood is \( \Lambda_{N1}(n) \), her neighborhood belief is

\[
\bar{q}(n) = P(\theta = 1 | a_{n-1})
\]

\[
= (1 - \epsilon_n - 1)[P(NS|a_{n-1})q(n, NS) + P(S|a_{n-1})q(n, S)]
\]

\[
+ \epsilon_n q(n),
\]

where \( q(n) \) (as in proposition (4.1)) can take two forms: \( q(n, 1) \) and \( q(n, 0) = 1 - q(n, 1) \) with

\[
q(n, 1) = (1 - \bar{q}(n - 1, 1)^2(3 - 2\bar{q}(n - 1, 1))) \frac{1 - \bar{q}(n - 1, 1)}{3 - 2\bar{q}(n - 1, 1)}
\]

\[
+ \bar{q}(n - 1, 1)^2(3 - 2\bar{q}(n - 1, 1)) \frac{2 - \bar{q}(n - 1, 1)}{3 - 2\bar{q}(n - 1, 1)}.
\]

This means, because of proposition (4.1),

\[
\bar{q}(n) = \frac{1}{2}.
\]

Also Let

\( q^*(n, NS) \): Belief that \( \theta = 1 \) as in proposition (4.2) when all agents up to \( n \) have neighborhood of type N2.

\( \bar{q}(n, NS) \): Belief that \( \theta = 1 \) as in proposition (4.2) when both \( n \) and \( (n - 1) \) have neighborhoods of type N2.
As shown in (4.2), \( q^*(n, NS) \) is increasing. This means the following.

\[
q(n - 1, NS) \leq q^*(n - 1, NS) \\
\Rightarrow \overline{q}(n, NS) \leq q^*(n, NS).
\]

So,

\[
q(n, NS) = \frac{3}{4} \varepsilon_n + \overline{q}(n, NS)(1 - \varepsilon_n) \\
\leq \frac{3}{4} \varepsilon_n + q^*(n, NS)(1 - \varepsilon_n) \\
\leq \frac{3}{4} \varepsilon_n + 1 - \varepsilon_n \\
< 1.
\]

and \( q(n, S) = 1 - q(n, NS) \).

Next, since \( q(n, NS) \geq \frac{1}{2} \), probability of taking the correct decision in this case evolves as follows.

\[
Pr(a_n^* = a_n^*(\theta)) = 1 - q(n, NS) + q(n, NS)^2 \\
\leq 1 - \frac{1}{4} \varepsilon_n (1 - \frac{1}{4} \varepsilon_n).
\] (10)

From (10), it follows that

\[
\varepsilon^* \neq 0 \Rightarrow \lim_{n \to \infty} Pr(a_n^* = a_n^*(\theta)) < 1
\]

and

\[
\varepsilon^* = 0 \Rightarrow \lim_{n \to \infty} Pr(a_n^* = a_n^*(\theta)) = 1.
\]

Thus \( \varepsilon^* = 0 \) is necessary and sufficient for social learning. \( \square \)

**Proof of Proposition (4.6).** Suppose agent \( n \) has neighborhood of type N2. As in proposition (4.5), I define the following.

\( q^*(n, NS) \): Belief that \( \theta = 1 \) as in proposition (4.2) when all agents up to \( n \) have neighborhood of type N2.

\( \overline{q}(n, NS) \): Belief that \( \theta = 1 \) as in proposition (4.2) when both \( n \) and \( (n - 1) \) have neighborhoods of type N2.
As shown in (4.2), \( q^*(n,NS) \) is increasing. This means the following.

\[
q(n - 1,NS) \leq q^*(n - 1,NS) \\
\Rightarrow \overline{q}(n,NS) \leq q^*(n,NS).
\]

So,

\[
q(n,NS) = \frac{3}{4} \varepsilon_n + \overline{q}(n,NS)(1 - \varepsilon_n) \\
\leq \frac{3}{4} \varepsilon_n + q^*(n,NS)(1 - \varepsilon_n) \\
\leq \frac{3}{4} \varepsilon_n + 1 - \varepsilon_n \\
< 1.
\]

and \( q(n,S) = 1 - q(n,NS) \).

Then as in (10),

\[
Pr(a_n^* = a_n^*(\theta)) \leq 1 - \frac{1}{4} \varepsilon_n(1 - \frac{1}{4} \varepsilon_n).
\]

Consider a \( \delta \) with \( 0 \leq \delta < \frac{1}{8} \). Next, define \( \overline{\varepsilon} = 1 - \sqrt{1 - 8\delta} \). Then

\[
\varepsilon^* \geq \overline{\varepsilon} \\
\Rightarrow \lim_{n \to \infty} Pr(a_n^* = a_n^*(\theta)) \leq 1 - \frac{1}{4} \varepsilon^*(1 - \frac{1}{4} \varepsilon^*) \\
\leq 1 - \frac{1}{4} \overline{\varepsilon}(1 - \frac{1}{4} \overline{\varepsilon}) \\
1 - \delta.
\]

Proof of Proposition (4.7). Suppose agent \( n \) has network structure of type N1. Further let \( n_1 \) be the number of agents preceding her whose network structure was of type N2. Obviously, \( n_1 \leq n \).
Then, if $a_{n-1} = 1$,
\[
q(n, 1) = Pr(\theta = 1|a_{n-1} = 1) \\
= Pr(NS|a_{n-1} = 1)q(n_1 + 1, NS) + Pr(S|a_{n-1} = 1)q(n_1 + 1, S) \\
= \frac{1}{2},
\]
where from proposition (4.2)
\[
q(n_1 + 1, NS) = q(n_1, NS)^2(2 - q(n_1, NS)) \\
+ (1 - q(n_1, NS)^2(3 - 2q(n_1, NS))) \frac{1 + q(n_1, NS)}{1 + 2q(n_1, NS)}.
\]
Hence, for agents with network structure N1, whatever the value of $q(n_1, NS)$,
\[
\lim_{n \to \infty} Pr(a_n^* = a_n^*(\theta)|N1) < 1.
\]
Next, suppose agent $n$ has the network structure of type N2. If agent $n - 1$ also has network structure N2, then
\[
q(n, NS) = q(n_1, NS)^2(2 - q(n_1, NS)) \\
+ (1 - q(n_1, NS)^2(3 - 2q(n_1, NS))) \frac{1 + q(n_1, NS)}{1 + 2q(n_1, NS)}.
\]
Else if agent $n - 1$ has network structure N1,
\[
q(n, NS) = q(n_1, NS).
\]
Probability of two consecutive agents being of type N2 is $(1 - \varepsilon)^2$. Hence, for agents with network structure N2, the probability of taking the correct decision
\[
Pr(a_n^* = a_n^*(\theta)|N2) = (1 - \varepsilon)^2(1 - q(n_1 + 1, NS) + q(n_1 + 1, NS)^2) \\
+ (1 - (1 - \varepsilon)^2)(1 - q(n_1, NS) + q(n_1, NS)^2).
\]
Let $n_2$ be the number occurrences of two consecutive agents of type N2 showing up before agent $n$. So, as $n \to \infty$, $n_2 \approx (1 - \varepsilon)^2 n \to \infty$. Hence,
\[
\lim_{n \to \infty} Pr(a_n^* = a_n^*(\theta)|N2) = \lim_{n \to \infty} (1 - q(n_2, NS) + q(n_2, NS)^2) \\
= 1.
\]
(Since from proposition (4.2) $q(n, NS) \to 1$ as $n \to \infty$).

Thus, agents with network structure N2 will have social learning. \qed
Proof of Lemma 4.3. Let $\Lambda_{N_k}(n)$ be the neighborhood of any agent $n$ under network structure $N_k$. Then, since $k > 1$,

$$\forall k, \Lambda_{N_2}(n) \subseteq \Lambda_{N_k}(n).$$

This means that

$$Pr\left(a^*_n = a^*_n(\theta) | N_k\right) = \max_{a_n} Pr\left(a_n = a^*_n(\theta) | N_k\right)$$

$$\geq \max_{a_n} Pr\left(a_n = a^*_n(\theta) | N_2\right)$$

$$= Pr\left(a^*_n = a^*_n(\theta) | N_2\right).$$
References


