Payoff-Irrelevant Traits in Asymmetric Coordination Games

1 Introduction

Traits that are irrelevant to people's capabilities are often present in everyday life, from physical traits like gender, race, appearance and height, personality traits like introversion and extroversion, to social traits like professional affiliations, political groups, religious groups. It is also often observed that people with different traits are treated differently on matters that are unrelated to those traits.

This paper establishes an evolutionary game theoretical model to find the driving force of such behavior. The methods applied in this paper are based on Kandori, Mailath and Rob (1999) (henceforth KMR). I find that discriminative behavior may increase efficiency within a certain population but from a broader perspective may result in loss of efficiency.

In section 2, I introduce a model with non-fixed traits. Then in section 3 I characterize the set of long run equilibria of this model and show that it is the set of Pareto efficient equilibria. In section 4 I introduce a model with fixed traits and two locations that are different in payoffs. I show that there exists a Darwinian update rule such that the set of long run equilbria is the set of separating equilibria, which results in inefficiency.

2 Non-fixed Trait Model

In this section, I introduce the model where traits are not fixed. A group of N myopic players are repeatedly matched to play a 2-by-2 asymmetric coordination game as follows, where (x, y) is the set of actions feasible and $a > b$, $d > c$, $a > d$.

	x	y
x	a, a	c, b
y	b, c	d, d

Each player has a observable trait, taking values in $\{1, 2\}$. The set of pure strategies is $\{xx, xy, yx, yy\}$ where xy means taking action x if the player is matched with a player of trait 1 and taking action y if the player is matched with a player of trait 2. Based on strategies and traits, the population can be divided into eight groups: $\{xx0, xy0, yx0, yy0, xx1, xy1, yx1, yy1\}$. Each period, a player is matched with each o the rest of the population exactly once.

In each period, the size of these eight groups are adjusted according to a deterministic update rule, $b\{\cdot\} : \mathbb{R}^8 \rightarrow \mathbb{R}^8$. Groups yielding better payoff grow in size. Note that both strategies and traits can be adjusted. There are multiple ways to understand adjustment of trait. For changeable trait like political opinions, hairstyle and etc., adjustment is fairly intuitive. For fixed traits like race, height and etc, adjustment can be considered as replacement. A player may be replaced by a new player with a different trait.

In each period, mutations take place at a probability 8ϵ to each player. Each mutated player becomes a member of one random group with equal probabilities. This mutation can be considered as the player experimenting or being replaced by a newcomer who has no information about the outcome of the previous period.

Let $z_t \in \mathbb{R}^8$ be the vector of the sizes of the eight groups. Then the procedure above yields the following set of nonlinear stochastic equations.

$$z_{t+1} = b(z_t) + x_t - y_t \tag{1}$$

where random vectors x_t and y_t have the following distribution:

each $y_t^i \sim Bin(b_i(z_t), 8\epsilon)$ and $x_t \sim Mult(\sum_{i=1}^{8} y_t^i, (\epsilon, \epsilon, \epsilon, \epsilon, \epsilon, \epsilon, \epsilon, \epsilon))$.

The dynamical system described by (1) defines a Markov chain on a finite state space, $Z = \{z \in \mathbb{N}^8 | \sum_i z^i = N\}$. The transition probabilities are given by $p_{z_1,z_2} = Prob(z_{t+1} = z_2 | z_t = z_1\}$ and $P = (p_{ij})$ is the Markov matrix. Since it is possible to mutate a state to any state, all p_{z_1,z_2}'s are positive. Then the Markov chain has a unique stationary distribution. This stationary distribution, $\mu(\epsilon)$, satisfies $\mu(\epsilon) = \mu(\epsilon)P$.

Here I refer to the definition of the limit distribution from KMR.

Definition. The limit distribution μ^* is defined by $\mu^* = lim_{\epsilon \to 0}\mu(\epsilon)$, if it exists. The set of long run equilibria is $C = \{z \in Z | \mu_z^* > 0\}$.

3 Long Run Equilibria

Since it is too complicated to explicitly to solve $\mu P = \mu$, I utilize the method due to Freidlin and Wenzell (1984), which is also used in KMR.

Definition. A z-tree h on a finite Z is a collection of ordered pairs of (or arrows between) elements of Z, denoted as $(i \to j)$, such that every state in $Z/\{z\}$ is the initial point of exactly one arrow and from any state in $Z/\{z\}$ there is a sequence of arrows leading to z. Let the set of all such z-trees be H_z.

Definition. Let $c_{z_1 z_2}$ be the speed of $p_{z_1 z_2}$ converges to zero as ϵ tends to zero, i.e. $p_{z_1 z_2} = O(\epsilon^{c_{z_1 z_2}})$.

Lemma. $c_{z_1 z_2} = |b(z_1) - z_2|/2$, where $|z| = \sum_i |z^i|$.

Proof. Consider the term of $p_{z_1 z_2}$ with the smallest power of ϵ. Q.E.D.

Theorem from KMR. Let $v_z = min_{h \in H_z} \sum_{(z_1 \to z_2) \in h} c_{z_1 z_2}$ be the minimal cost of all z-trees of a state z. Then $C = Argmin_{z \in Z} v_z$.

In this section, I characterize the set of long run equilibria, with certain assumptions on the update rule, $b(\cdot)$ and payoffs.

First I define the basin of attraction.

Definition. The basin of attraction of a state z is $B(z) = \{i : lim_{k\to\infty} b^k(i) = z\}$.

Assumption. (1) if a group's payoff is strictly larger than the population median, its size strictly increases if possible. If a group's payoff is strictly smaller than the population median, its size strictly decreases if possible.

(2) If a group has maximum payoff in population, its size does not decrease. If a group has minimum payoff in population, its size does not increase.

(3) If $i \in B(z)$, then $|i - z| > |b(i) - z|$.

(4) If $B(z)$ is not empty, then there must exist $z' \in B(z)$ such that $\frac{1}{2}|z - z'| = 1$.

(5) At least one of $\frac{(d-b)(d-c)n-(a-c)(a-d)}{a^2-ab-ac-bc+2bd+2cd-2d^2}$, $\frac{(a-d)(a+c-2d-cn+dn)}{a^2-ab-ac-bc+2bd+2cd-2d^2}$, $\frac{(a-d)(2d-2a+(a-b)n)}{a^2-ab-ac-bc+2bd+2cd-2d^2}$ is not an integer.

(6) At least one of $\frac{(a-b)(a-c)n-(a-d)(b-d)}{2a^2-2ab-2ac+bc+bd+cd-d^2}$, $\frac{(a-d)(d-c)n+(2a-2d)(a-d)}{2a^2-2ab-2ac+bc+bd+cd-d^2}$, $\frac{(a-b)(a-d)n-(a-d)(2a-b-d)}{2a^2-2ab-2ac+bc+bd+cd-d^2}$ is not an integer.

Theorem 1. If assumption (1) to (6) are satisfied, then

$$C = \{z : z^{xx0} + z^{xy0} = N\} \cup \{z : z^{xx0} + z^{xx1} = N\} \cup \{z : z^{yx1} + z^{xx1} = N\}.$$

Proof.

First, I solve for the set of the steady states, i.e. $Z^* = \{z|b(z) = z\}$. There are many elements in Z^* and Z^* can be partitioned into four subsets, I_1 to I_4. I examine them one by one.

$I_1 = \{z : z^{xx0} + z^{xy0} = N\} \cup \{z : z^{xx0} + z^{xx1} = N\} \cup \{z : z^{yx1} + z^{xx1} = N\}$.

$I_2 = \{z : z^{yx0} + z^{yy0} = N, z_{yx0} \leq \frac{d-c}{a-c}\} \cup \{z : z^{yy0} + z^{yy1} = N\} \cup \{z : z^{xy1} + z^{yy1} = N, z_{xy1} \leq \frac{d-c}{a-c}\}$

$I_3 = \{z : z_{xy0} = z_{yx1} = \frac{N}{2}\} \cup \{z : z_{yx0} = z_{xy1} = \frac{N}{2}\}$

A detailed characterization of I_4 is available in the appendix.

First, I show that for any $z_1, z_2 \in I_1$, $v_{z_1} = v_{z_2}$. Take any z-tree h of z_1, and make the following change: $z' \to b(z')$ for $z' \in B(I_1) - I_1$, $z' \to$ *immediate neighbour* (towards z) for

$z^{'} \in I_1 - z_2$. h becomes a z-tree of z_2. Note that $c_{z',b(z')} = 0$ and any arrow from a steady state costs at least one. Therefore this change does not increase the cost of the z-tree. Hence all $z \in I_1$ has equal v_z. Similarly, for any $z_1, z_2 \in I_2$, $v_{z_1} = v_{z_2}$.

Now I show that $I_3 \cap C = \emptyset$. Take $z_0 = (z_{xy0} = \frac{N}{2},\ z_{yx1} = \frac{N}{2})$ and consider $z_1 = (z_{xy0} = \frac{N}{2} + 1,\ z_{yx1} = \frac{N}{2} - 1)$. By the assumptions, $z_1 \in B(\{z|z_{xy0} = N\}) \subset B(I_1)$. Notice that $\forall z^{'} \in I_1 s.t\ |z - z^{'}| = 1\}, z^{'} \in B(I_1)$.

If there exists z^* whose arrow costs at least 2, fix this z^*. Otherwise, there must exists z^*, z' such that $z^* \to z'$ and $z' \in B(I_1) - I_1$ since the minimum cost of an arrow from any $z \in I_1$ to any state $z'' \notin B(I_1)$ is at least 2. Now make the following change:$z_0 \to z_1$, $z \to b(z)$ for $z \in B(I_1) - I_1$, $z \to$ *immediate neighbour* (towards z^*) for $z \in I_1$. Then the original z-tree of z_0 becomes a z-tree of z^*, and its cost is lowered by at least 1. Therefore $z_0 \notin C$. Similarly, $(z_{yx0} = \frac{N}{2}, z_{xy1} = \frac{N}{2}) \notin C$.

Now I show that $I_2 \cap C = \emptyset$. Take $z_e = (z_{yx0} = \left\lfloor \frac{a-d+(d-c)n}{a-c} \right\rfloor, z_{yy0} = N - \left\lfloor \frac{a-d+(d-c)n}{a-c} \right\rfloor) \in I_2$ and consider $z'_e = (z_{yx0} = \left\lfloor \frac{a-d+(d-c)n}{a-c} \right\rfloor + 1, z_{yy0} = N - \left\lfloor \frac{a-d+(d-c)n}{a-c} \right\rfloor - 1)$. Then $z'_e \in B(z_3)$ where $z_3 = (z_{xx1} = N)$. Then by similar procedure as shown above, I can construct a z-tree of z_3 with strictly lower cost. Since all $z \in I_2$ have equal v_z, $\forall z_1 \in I_1, z_2 \in I_2$, $v_{z_1} < v_{z_1}$ thus $I_2 \cap C = \emptyset$.

For all states in I_4, by the assumptions, each has a singleton basin of attraction. Fix a state $z_s \in I_4$ and let z'_s be a neighbour of z_s. Now $z'_s \in B(I_1) \cup B(I_2) \cup B(z_0) \cup B(z'_0)$ since each state in I_4 has a singleton basin of attraction. By similar analysis, $v_{z'_x} > min\{v_z|z \in I_1 \cup I_2 \cup I_3\} = v_z|z \in I_1$. Therefore $I_4 \cap C = \emptyset$.

Since C is nonempty and $\forall z_1, z_2 \in I_1$, $v_{z_1} = v_{z_2}$, I have

$$C = I_1 = \{z : z^{xx0} + z^{xy0} = N\} \cup \{z : z^{xx0} + z^{xx1} = N\} \cup \{z : z^{yx1} + z^{xx1} = N\}.$$

Q.E.D.

Note that in KMR, the long run equilibrium is the risk dominant one. In this model, the set of long run equilibria is the set of Pareto efficient equilibria. Even though the existence of payoff-irrelevant trait leads to efficiency, in majority of long run equilibria, one trait is completely been eliminated. In reality, this can be represented by, for example, groups and organizations eventually sharing the same political opinions, dress code or even race (people of different race are edged out and replaced).

4 Fixed Trait Model

In this section, I introduce a variation of the model of non-fixed trait. A group of N players reside at two locations, $\{0, 1\}$. Each period, a player is matched with each of the players (including self) at the same location for exactly once, to play the same asymmetric coordination as described above. I include interaction with self to avoid the situation where a player is alone at one location and his payoff becomes zero since there is no other player at that location. Each player gets the average payoff of all the games he plays in each period. Each player has an observable trait and this trait cannot be adjusted or mutated. Each period, adjustment and mutation occurs to players' strategies and locations. All players at location 2 receive payoffs discounted by δ.

Definition. An update rule is defined 'Darwinian' if under this update rule, if a group has maximum payoff in population, its size does not decrease, and if a group has minimum payoff in population, its size does not increase.

Theorem 2. If $\frac{ab-b^2-bc+bd-ad+ac}{a(a-b-c+d)} > \delta$, there exists a Darwinian update rule such that in all long run equilibria, players of different traits are completely separated into the two locations.

Proof. Consider the following dynamics. Starting from any separating state, all players with trait 0 reside at location 0 and all players of trait 1 reside at location 1 and all players choose x when matched to another player with the same trait. Whenever $z_{xy0,0} < \frac{a(1-\delta)}{a-c}$, players of trait 1 will start to move to location 0. Consider the update rule that facilities the following dynamics.

For any $s \leq \frac{a(1-\delta)}{a-c}$,

$$(z_{xx0,0} = \frac{N}{2}(1 - \frac{a(1-\delta)}{a-c} + s), z_{xy0,0} = \frac{N}{2} \cdot (\frac{a(1-\delta)}{a-c} - s), z_{xx1,1} = \frac{N}{2})$$

$$\to (z_{xx0,0} = \frac{N}{2}(1 - \frac{a(1-\delta)}{a-c} + s), z_{xy0,0} = \frac{N}{2} \cdot (\frac{a(1-\delta)}{a-c} - s), z_{xx1,0} = \frac{N}{2} \cdot \frac{d-c}{a-b-c+d}, z_{xx1,1} = \frac{N}{2} \cdot \frac{a-b}{a-b-c+d})$$

$$\to (z_{xx0,0} = \frac{N}{2}(1 - \frac{a(1-\delta)}{a-c} + s), z_{xy0,0} = \frac{N}{2} \cdot (\frac{a(1-\delta)}{a-c} - s), z_{xx1,0} = \frac{N}{2} \cdot \frac{d-c}{a-b-c+d}, z_{yx1,0} = \frac{N}{2} \cdot \frac{a-b}{a-b-c+d})$$

$$\to (z_{xy0,0} = \frac{N}{2}, z_{xx1,0} = \frac{N}{2} \cdot \frac{d-c}{a-b-c+d}, z_{yx1,0} = \frac{N}{2} \cdot \frac{a-b}{a-b-c+d}$$

$$\to (z_{xy0,0} = \frac{N}{2}, z_{yx1,0} = \frac{N}{2})$$

$$\to (z_{xy0,0} = \frac{N}{2}, z_{xx1,1} = \frac{N}{2})$$

By similar analysis as in the proof of theorem 1, all separating steady states z's have equal v_z. From the analysis above, the set of such steady states is absorbing and therefore those are the only long run equilibria. Q.E.D.

5 Conclusion

This paper explores the influence of payoff-irrelevant traits on the evolutionary outcome of asymmetric coordination games. I show that with non-fixed traits, Pareto efficiency is always achieved and in majority of the long run equilibria, one trait is eliminated from population. Here a population can be represented by simply a group or an organization, for example a company. It is often observed that the employees in one firm can be rather united on political opinions and some firms have a nearly unified racial makeup.

This paper also shows that with fixed traits and two different locations, there can be situations where separation is always present, in spite of efficiency loss. Such behavior can be often observed, such as black and white residential communities.

Reference

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