Effective Field Theories of Topological Insulators

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Motivation

- To derive an effective field theory of topological phases
- Hydrodynamic low energy theory for the charge degrees of freedom as in the FQHE
- Distinction between topologically protected response and topological phase: when is a topological phase?
- Candidates: in 2D Chern-Simons; in 3D: axion+BF theory
- How are topological insulators related to topological phases?
- 2D vs 3D; $\mathbb{Z}$ vs $\mathbb{Z}_2$
- Fractionalization
- Exciting new area for numerical simulations at strong coupling

What is a topological phase?

- Fractional Quantum Hall fluids of 2DEG in high magnetic fields
- Deconfined phases of discrete gauge theories
- Topological phases do not break any symmetries
- Natural representation in terms of gauge theories
- Gapped bulk excitations and gapless edge states (not always)
- Bulk excitations: vortices of the incompressible fluid and carry fractional charge (charged fluid) and fractional statistics
- On a closed 2D manifold with g handles: ground state degeneracy k^g where k is an integer that depends on the topological phase
- Effective field theory: topological QFT, e.g Chern-Simons, BF theory

Hydrodynamic Theory of the FQH States
(Fröhlich and Zee, Wen)

- Example: Laughlin FQH states @ filling fraction $\nu=1/k$
- Gapped states with a conserved charge current

$$\partial_\mu j^\mu = 0 \qquad \Longrightarrow j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \mathcal{A}_\lambda$$

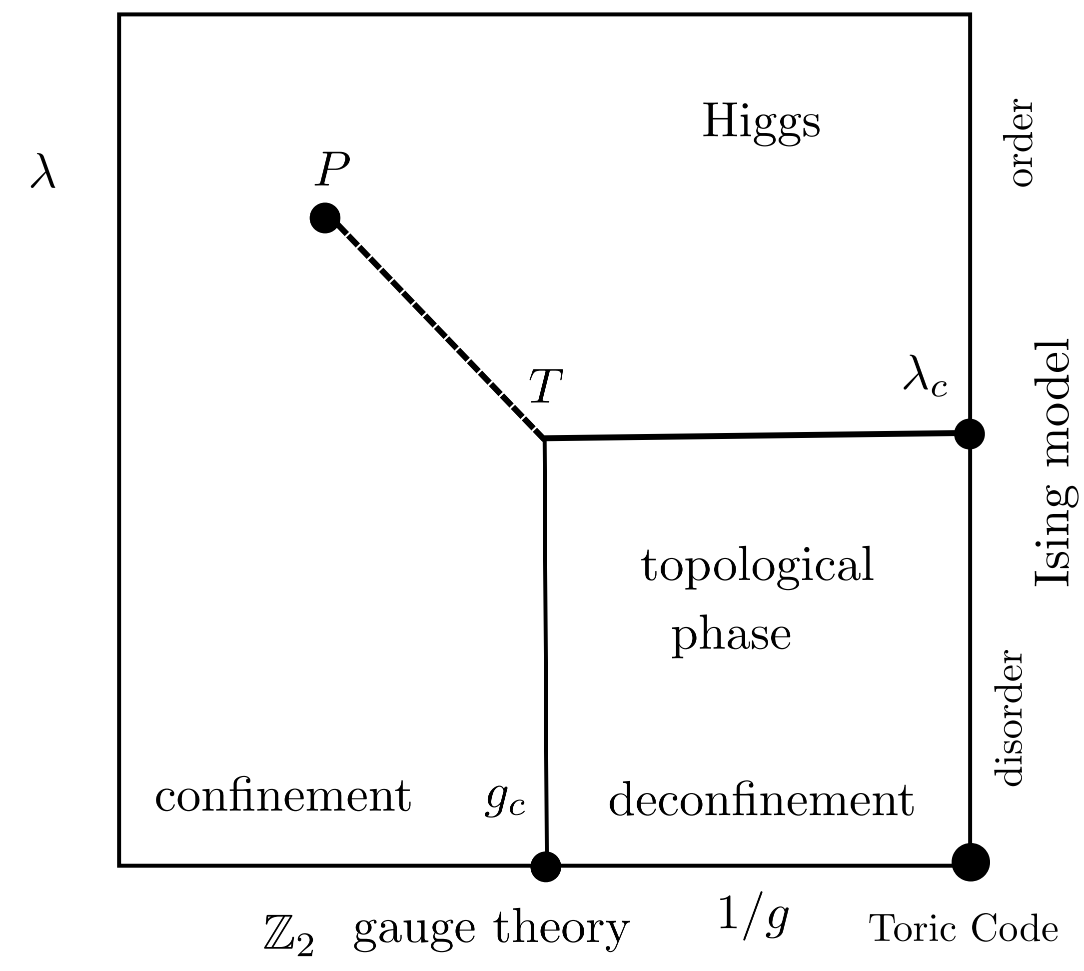
$$\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \Phi \qquad \Longrightarrow j_\mu \to j_\mu$$

- Gapped fluid: effective action is local and gauge invariant
- Broken time reversal invariance and parity (magnetic field)
- Effective action: Chern-Simons gauge theory

$$S_{\text{eff}}[j_\mu] = \frac{k}{4\pi}\int d^3x \epsilon^{\mu\nu\lambda}\mathcal{A}_\mu \partial_\nu \mathcal{A}_\lambda - \frac{e}{2\pi}\int d^3x A_\mu^{\text{ext}}\epsilon^{\mu\nu\lambda}\partial_\nu \mathcal{A}_\lambda$$

Discrete Gauge Theories

- Discrete gauge theories (e.g. $\mathbb{Z}_2$ gauge theory) have confining and deconfined ("free charge") phases.
- The topological sectors of deconfined phases are described by the algebra of Wilson and 't Hooft loops on non-contractible cycles on the torus
- d=2 spacial torus they have a 4-fold degenerate vacuum
- Fractionalized excitations
- Equivalent to Kitaev's "Toric Code" states
- Described by a BF theory ("double" Chern-Simons)



Topological Insulators

- The electronic states of solids are Bloch states $|\mathbf{u_n(k)}\rangle$ which form energy bands labeled by a band index n and (quasi) momentum **k**, defined on a Brillouin zone (BZ)
- The BZ is a torus (2-torus in 2D, 3-torus in 3D)
- Insulator: integer number of bands are filled
- Topological insulator: the bands have a non-trivial topological structure
- In some cases the bands have an integer-valued topological invariant: the *Chern number*
- In other cases there is a $\mathbb{Z}_2$ topological invariant
- K theory and Bott periodicity classification of topological insulators (A. Kitaev; S. Ryu, A. Schnyder, and A. Ludwig) (2009)

Topology of electronic states

- Prototype: IQH on 2D lattices in magnetic fields (Thouless, Kohmoto, Nightingale, den Nijs) (TKNN) (1982)
- Bloch states $|\mathbf{u_n(k)}\rangle$ are multivalued functions on the BZ (torus) and must be defined on patches
- Bloch states are characterized by the Chern number C_n of the band (M. Berry; B. Simon) (1982)

$$\mathcal{A}_j^{(n)}(\mathbf{k}) = i\left\langle u_n(\mathbf{k})\middle|\nabla_{\mathbf{k_j}}\middle|\mathbf{u_n}(\mathbf{k})\right\rangle$$

- Berry connection

$$\int_{BZ} d^2k\, \mathcal{F}^{(n)} = \oint_\Gamma d\mathbf{k}\cdot\mathcal{A}^{(\mathbf{n})}(\mathbf{k})$$

- Berry curvature

$$\mathcal{C}_n = \frac{1}{2\pi}\oint_\Gamma d\mathbf{k}\cdot\mathcal{A}(\mathbf{k})$$

- Chern number

Hall conductance is a topological invariant (TKNN, 1983)

$$\sigma_{xy}^{(n)} = \frac{e^2}{2\pi\hbar}\mathcal{C}^{(n)}$$

Dirac Fermions and Band Crossings

- Band crossings occur in a number of 3D materials such as Bi_2Se_3.
- The conduction and valence bands can become (nearly) degenerate at a finite number of points of the BZ, e.g. $\mathbf{k}=0$
- The degeneracy is protected by symmetry, e.g. time reversal
- Spin-orbit interaction is large in many materials
- Close to the degeneracy point the electronic states are represented as Dirac 4-spinors: the two band indices and the two angular momentum projections
- Locally in momentum space this is a Dirac fermion and the effective Hamiltonian is the Dirac Hamiltonian

3D Z_2 Topological Insulator

(Qi, Hughes, Zhang; Fu, Kane and Mele) (2007)

- A good example of a Z_2 TI is Bi_2Se_3
- A simple lattice model (using the 4 x 4 Dirac matrices) ("Wilson fermions")

$$H = \int_{BZ} \frac{d^3p}{(2\pi)^3}\; \psi^\dagger(\mathbf{p}) \left(\sin(\mathbf{p}) \cdot \alpha + M(\mathbf{p})\beta \right)(\mathbf{p})\; \psi(\mathbf{p})$$

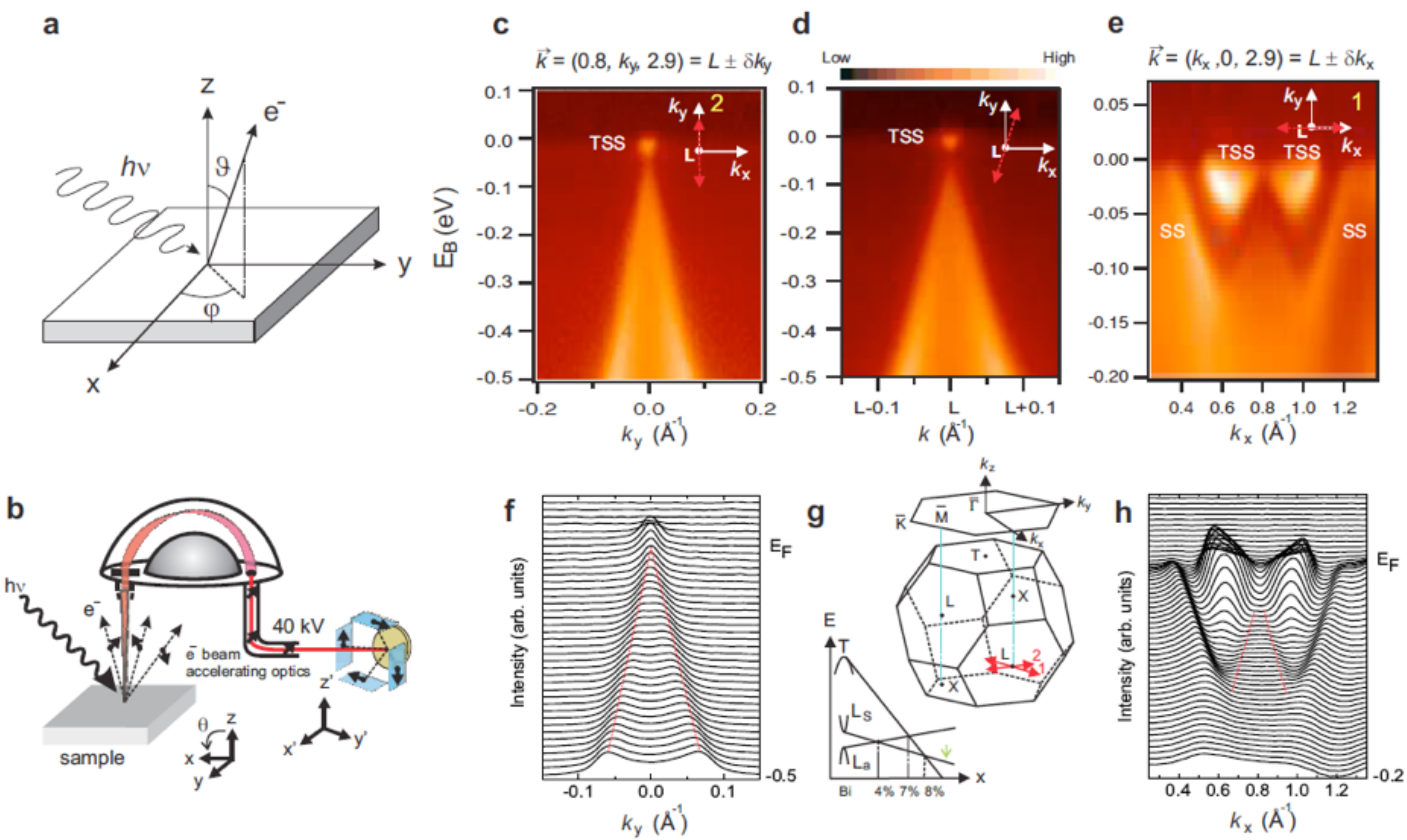
$$M(\mathbf{p}) = M + \cos p_x + \cos p_y + \cos p_z - 3$$

- 0< M < 2: negative energy states (near k=0) have negative parity, positive energy states have positive parity
- Parity of the states at the time-reversal-invariant points (π,π,π), (π,π,0), and (π,0,0) are reversed relative to (0,0,0)
- The product of the parities is a topological invariant since it does not change under smooth changes of the Hamiltonian

Topologically Protected Surface Weyl Fermions

- The surface of a 3D Z_2 TI behaves as the locus of a topological domain wall
- Parity Anomaly through a Callan-Harvey effect (1985)
- Early theory by Boyanovsky, Dagotto and EF (1986)
- The 3D $\mathbb{Z}_2$ TI has two-component massless Dirac (Weyl) fermions
- The fermions have opposite chirality (helicity) on the two opposite surfaces
- The helicity of the surface Weyl states has been measured in polarized angle-resolved photoemission experiments (ARPES) (Hasan, Shen...) (2010) and the Dirac spectrum with STM (Kapitulnik, Yazdani…) (2013)

ARPES Experiments in Bi_2Se_3 (Hsieh et al) (2011)



Bosonization

- Use bosonization to derive the effective field theory of topological insulators
- It is equivalent to bosonization in D=1+1 as an operator identity for free gapless fermions
- For D>1+1 we will get an effective hydrodynamic field theory for gapped fermions
- Extension to interacting fermions and fractionalization

$$Z[A^{\mathrm{ex}}] = \int \mathcal{D}\left[\bar{\psi}, \psi\right] \exp\left(iS_F[\bar{\psi}, \psi, A^{\mathrm{ex}}]\right)$$

$$\langle j^{\mu_1}(x_1) j^{\mu_2}(x_2) \cdots \rangle = \frac{1}{i} \frac{\delta}{\delta A^{\mathrm{ex}}_{\mu_1}(x_1)} \frac{1}{i} \frac{\delta}{\delta A^{\mathrm{ex}}_{\mu_2}(x_2)} \cdots \ln Z[A^{\mathrm{ex}}]$$

Use gauge invariance of the fermion path integral:

shift A^{ex} to A^{ex} + a, where a is a gauge transformation:

$$f_{\mu\nu}[a] = 0 \qquad\qquad Z[A^{\rm ex} + a] = Z[A^{\rm ex}].$$

$$Z[A^{\rm ex}] = \int \mathcal{D}[a]_{\rm pure} Z[A^{\rm ex} + a]$$

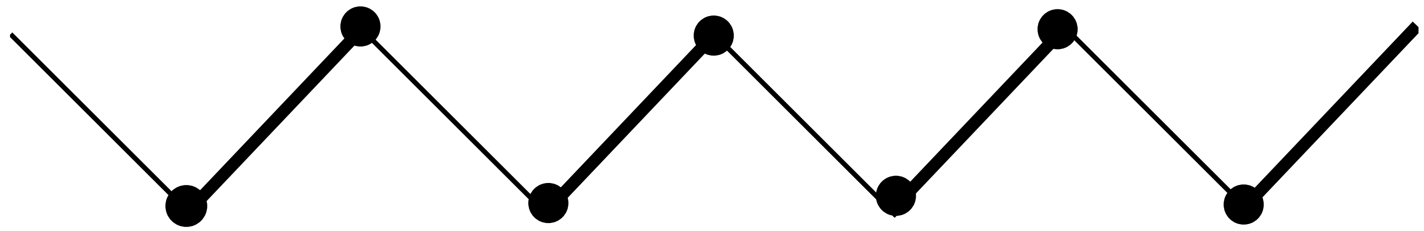
$$Z[A^{\rm ex}] = \int \mathcal{D}[a,b] Z[a] \times \exp\Big(-\frac{i}{2}\int d^D x\, b_{\mu\nu\dots}\epsilon^{\mu\nu\cdots\alpha\beta}\left(f_{\alpha\beta}[a] - f_{\alpha\beta}[A^{\rm ex}]\right)\Big)$$

$$\langle j^{\mu_1}(x_1) j^{\mu_2}(x_2)\cdots\rangle = \langle \epsilon^{\mu_1\nu_1\lambda_1\cdots}\partial_{\nu_1} b_{\lambda_1\dots}(x_1)\epsilon^{\mu_2\nu_2\lambda_2\cdots}\partial_{\nu_2} b_{\lambda_2\dots}(x_2)\cdots\rangle$$

$$j^\mu(x) \quad \equiv \quad \epsilon^{\mu\nu\lambda\rho\cdots}\partial_\nu b_{\lambda\rho\dots}(x) \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0$$

- This procedure is meaningful only if the effective action is local (Schaposnik and EF, 1994; Burgess and Quevedo, 1994)
- This works in 1+1 dimensions for gapless relativistic fermions
- For D>1+1 it works only if there is a finite energy gap
- This leads to a hydrodynamic description
- For systems with a Fermi surface one obtains the Landau theory of the Fermi liquid

Example: Polyacetylene



- Fermions in d=1 with a spontaneously broken translation symmetry: broken chiral symmetry

$$\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \to e^{i\sigma_3\theta} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \qquad \rho(x) \to \rho(x+a) \Rightarrow \theta = k_F a$$

Topological invariant $$\nu = \frac{\theta(+\infty) - \theta(-\infty)}{2\pi}$$

Charge conjugation (particle-hole) $$\theta = \nu\pi \mod 2\pi$$

$$Z[A^{\text{ex}}] = \int \mathcal{D}[a,b] \exp\left(i \int d^D x \mathcal{L} \right)$$

$$\mathcal{L} = -b\epsilon^{\mu\nu}\partial_\mu(a_\nu - A_\nu^{\text{ex}}) + \frac{\theta}{2\pi}\epsilon^{\mu\nu}\partial_\mu a_\nu + \cdots$$

D=2+1 Chern Insulator

- Free fermions with broken time reversal invariance: integer quantum Hall states and the quantum anomalous Hall state
- The states are characterized by a topological invariant, the Chern number *Ch*
- In this case we obtain

$$\mathcal{L} = -b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A_\lambda^{\rm ex}) + \frac{\mathsf{Ch}}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda .$$

- The first term is the BF Lagrangian
- The hydrodynamic field b_μ couples to flux tubes
- The statistical gauge field a_μ couples to quasiparticle worldlines

Quantized Hall conductance $\sigma_{xy} = Ch \, \frac{e^2}{h}$

3D Topological Insulator

- Massive relativistic fermions with a conserved U(1) charge.
- Topological invariant: the winding number

$$\mathcal{L} = -b_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda(a_\rho - A^{\rm ex}_\rho) + \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu \partial_\lambda a_\rho - \frac{1}{4\pi^2 g^2}\partial_\mu a_\nu \partial^\mu a^\nu + \cdots$$

The field $b_{\mu\nu}$ is an antisymmetric tensor field

If time-reversal (particle-hole) is imposed, the topological class is $\mathbb{Z}_2$

$$\text{with } \theta = \nu\pi \mod 2\pi$$

The effective action for the external gauge field has an axion term (Qi, Hughes, Zhang, 2009)

Physical Picture

The effective field theory we found (without the topological term) has the same form as the topological field theory of a d=2 gapped superconductor (Hansson, Oganesyan and Sondhi, 2004)

$$\mathcal{L} = -\frac{2k}{4\pi} b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\sigma - A^{\text{ex}}_\lambda) + \cdots$$

- In our case k=1, in the SC k=2.
- for k>1 this is a fractionalized state
- In the case of the SC fractionalization follows from pairing: the Bogoliubov quasiparticles have spin but no charge
- The k=2 BF theory also describes the topological limit of the deconfined phase of the $\mathbb{Z}_2$ gauge theory

Topological Insulator in 2+1 Dimensions

$$\mathcal{L} = -\frac{2k}{4\pi} b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A^{\text{ex}}_\lambda) + \frac{Ch}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$
$$= \frac{K_{ij}}{4\pi} \alpha^i_\mu \epsilon^{\mu\nu\lambda} \partial_\nu \alpha^j_\lambda + \frac{k}{2\pi} b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu A^{\text{ex}}_\lambda$$

$$K = \begin{pmatrix} 0 & -k \\ -k & Ch \end{pmatrix} \qquad (\alpha^1_\mu, \alpha^2_\mu) = (b_\mu, a_\mu)$$

- The effective field theory has the K-matrix form (as in the fractional quantum Hall fluids) (Wen and Zee) with a degeneracy k^2 on the torus
- A theory with k>1 is a fractionalized topological Chern insulator
- To derive this state once has to resort to flux attachment or parton constructions

Electromagnetic Response

The effective action for the external gauge field is

$$S_{\rm eff}[A^{\rm ex}] = \frac{1}{8\pi^2} \int d^4x\, \theta \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu^{\rm ex} \partial_\rho A_\sigma^{\rm ex}$$

If the bulk mass gap is allowed to have a chiral twist, one finds a current

$$j^\mu = \epsilon^{\mu\nu\lambda\rho} \partial_\nu b_{\lambda\rho} = \frac{1}{4\pi^2} \epsilon^{\mu\nu\lambda\rho} \partial_\nu \left(\theta \partial_\lambda A_\rho^{\rm ex}\right)$$

If the chiral angle has a jump from 0 to π at the surface one obtains a surface Hall conductivity with 1/2 of the quantized value

$$\sigma_{xy} = \frac{1}{4\pi} = \frac{e^2}{2h}$$

Fractionalized $\mathbb{Z}_2$ Topological Insulator

- Fractionalized U(1) TI in 2D: FQH state at zero field
- It is found in models of 2D TI's with "flat bands" (Sun et al (2011), Bernevig et al (2011) , Neupert et al (2011))
- Not much is known on a 3D fractionalized $\mathbb{Z}_2$ TI
- One can obtain an effective field theory using a parton construction (flux attachment does not work in 3D)
- It requires having at least two species of a TI and finding a state in which the species are "paired"
- In this case the difference of the two hydrodynamic fields becomes massive (Higgs mechanism)
- In the low-energy limit the different hydrodynamic fields become identified with each other

Effective Field Theory with k>1

In the simplest case we get a theory at a higher level k

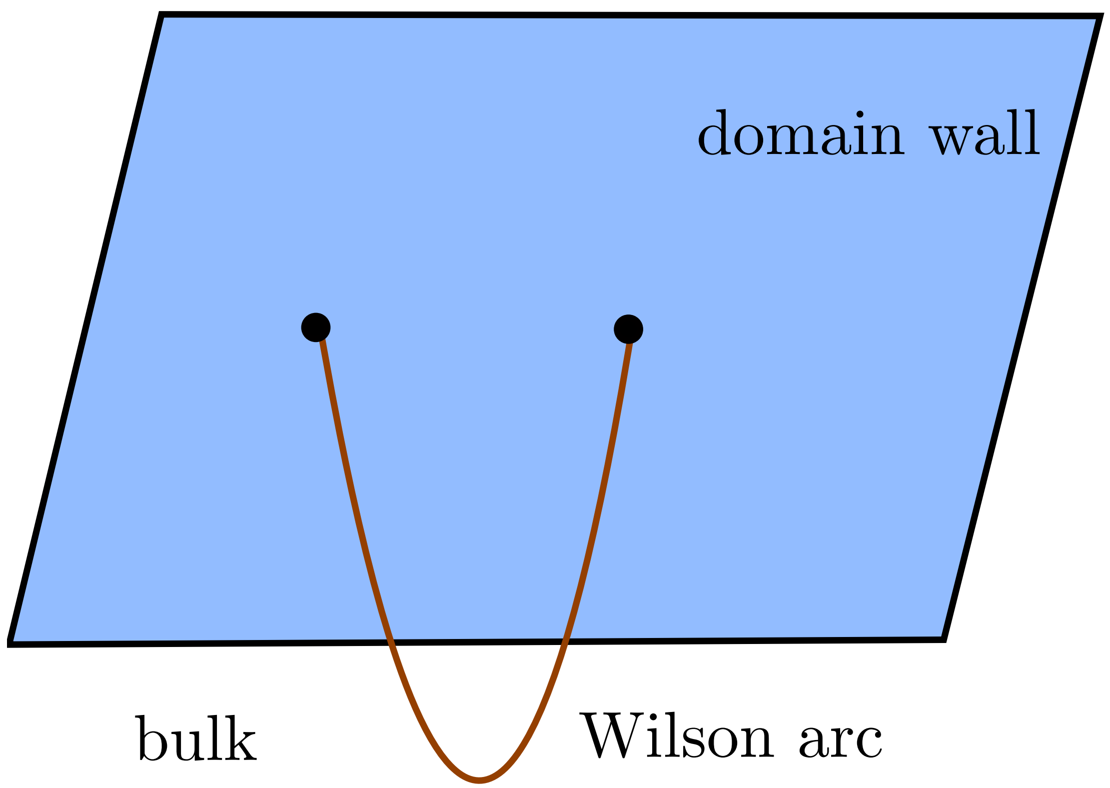
$$\mathcal{L} = -\,\epsilon^{\mu\nu\lambda\rho} b_{\mu\nu} \sum_{I=1}^{k} \partial_\lambda a^I_\rho - e\,\epsilon^{\mu\nu\lambda\rho} \partial_\nu b_{\lambda\rho} A^{\text{ex}}_\mu$$

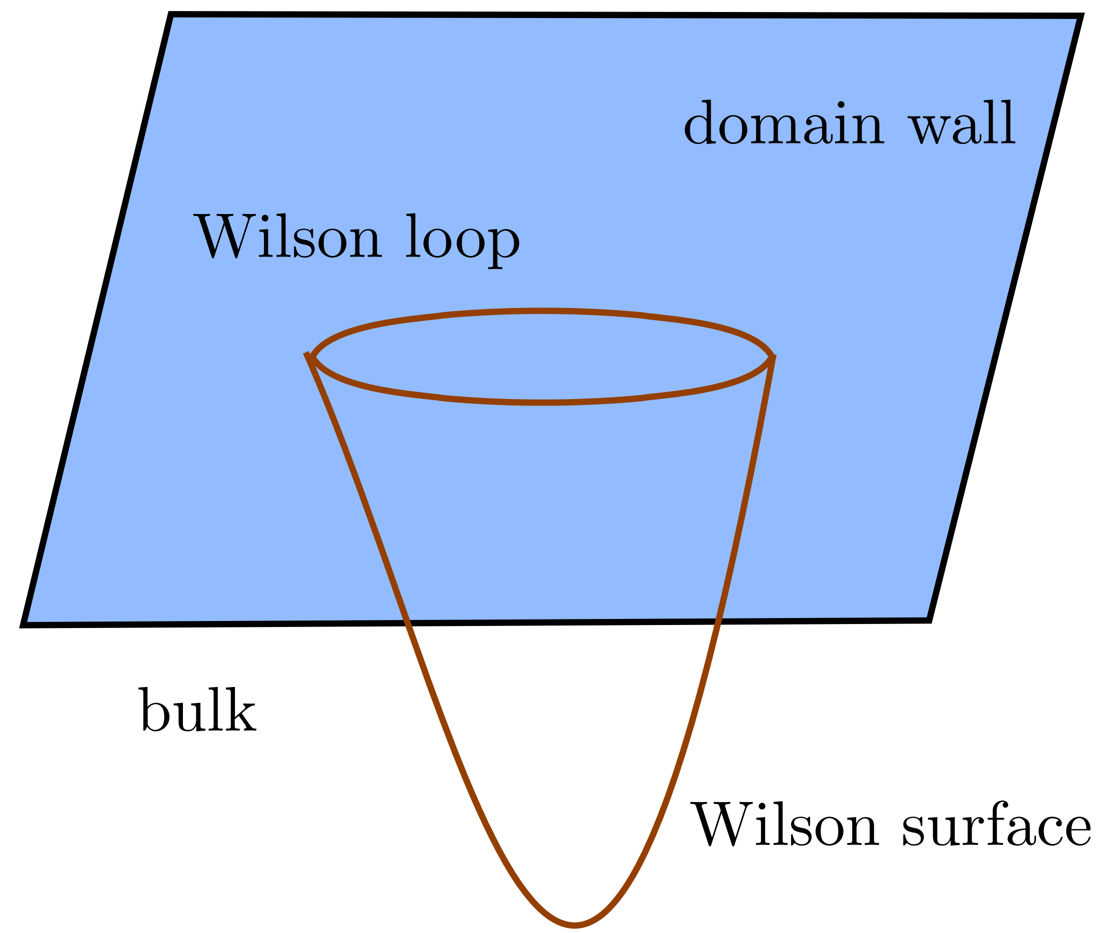
$$+\frac{\theta}{8\pi^2} \sum_{I=1}^{k} \epsilon^{\mu\nu\lambda\rho} \partial_\mu a^I_\nu \partial_\lambda a^I_\rho - \frac{1}{4g^2} \sum_{I=1}^{k} (f^I_{\mu\nu})^2$$

Projection: $a^I_\mu \equiv a_\mu \quad \Rightarrow \quad \mathcal{L} = \frac{\theta}{8\pi^2 k} \epsilon^{\mu\nu\lambda\rho} \partial_\mu A^{\text{ex}}_\nu \partial_\lambda A^{\text{ex}}_\rho$

- This system has a fractional surface Hall conductance $\sigma_{xy}=e^2/(2kh)$
- Fractional Callan-Harvey effect
- On opposite surfaces the Hall conductance has opposite signs
- surface gapped phases and quantum phase transitions

Gapped Surface Phases





- Fractional charge and fractional statistics on the surface are endpoints of Wilson arcs of the bulk
- Fractionalized bulk extended excitations (Wilson surfaces coupled to the BF antisymmetric tensor fields) are Wilson loops of the surface states

Quantum Phase Transitions and Modular Symmetry

- Gapped topological phases with broken time reversal invariance described by Chern-Simons gauge theory
- The bulk theory has an extended duality (modular $SL(2,\mathbb{Z})$) symmetry of the coupling $z=(\theta/2\pi)+i4\pi/g^2$ (Witten, 1995) which induces a modular symmetry on the boundary degrees of freedom (S. Kivelson and EF, 1996)
- Quantum critical points are *fixed points* of subgroups of $SL(2,\mathbb{Z})$, with finite universal values of σ_{xx} and σ_{xy}
- Similarity of the quantum Hall plateau transitions but not driven by disorder

Conclusions and Outlook

- New arena in which to find new topological phases of matter
- Unlike the FQH fluids of the 2DEG in magnetic fields the fractionalized TI phases require non-trivial interactions
- The interactions are neither too weak nor too strong
- Important arena for numerical simulations
- Strongly coupled "domain-wall fermions"
- They have been realized in relatively simple models but, so far, they have not been realized in experiment
- 2D fractionalized TI's are extensions of the FQH fluids
- The 3D fractionalized TI have novel features
- Structure of their wave functions of fractionalized TI's?
- Classification?