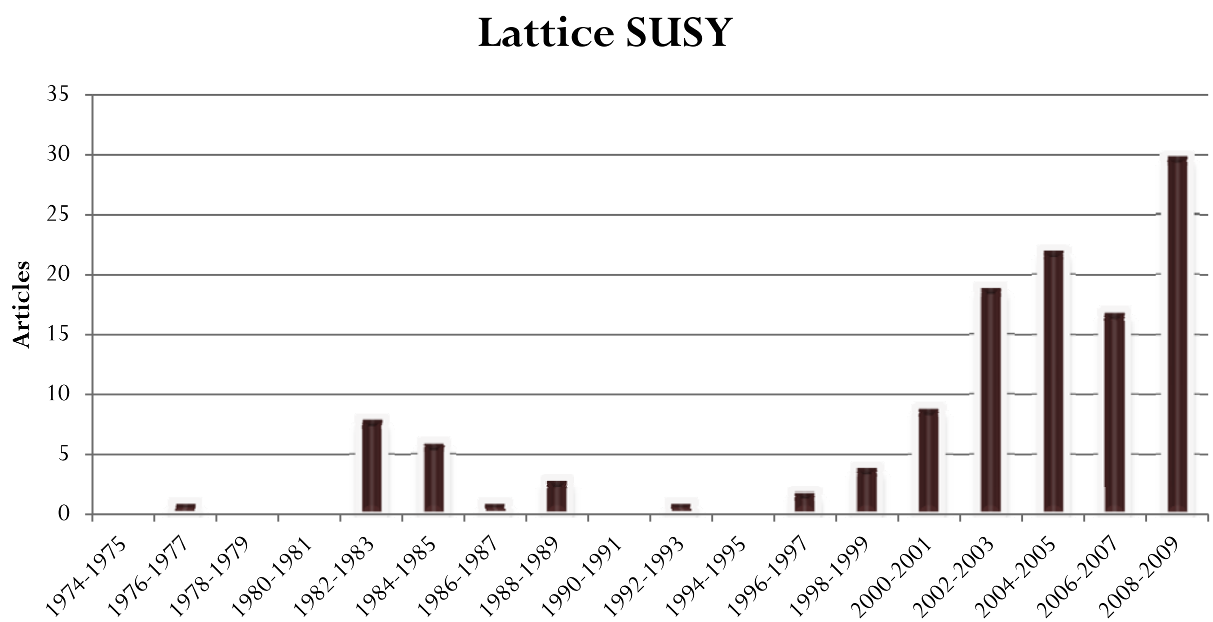
Lattice N=4 SYM

Joel Giedt (RPI)

in collaboration with Simon Catterall, Anosh Joseph, Eric Dzienkowski, Robert Wells, David Schaich, Tom DeGrand, Poul Damgaard

The lattice SUSY story

Computational resources now allow dynamical fermions



Lattice formulations that reduce/eliminate fine-tuning

Why not more?

The lattice SUSY problem

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu$$

- P_μ generator of infinitesmal translations.
- Broken on lattice.
- Only discrete subgroup preserved.
- with lattice spacing a

$$x \to T_\mu x = x + a\hat{\mu}$$

Failure of Leibnitz rule

- So why not just change the algebra to

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\tfrac{1}{a}(T_\mu - 1) \equiv 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu$$

- Notice that

$$\nabla_\mu\phi(x) = \tfrac{1}{a}[\phi(x+a\hat\mu) - \phi(x)] = \partial_\mu\phi(x) + \tfrac{a}{2}\partial_\mu^2\phi(x) + \mathcal{O}(a^2)$$

- Problem: essential difference between these (∂_μ and ∇_μ) at finite lattice spacing.
- The Leibnitz rule.

$$\nabla_\mu[\phi(x)\chi(x)] = \tfrac{1}{a}[\phi(x+a\hat\mu)\chi(x+a\hat\mu) - \phi(x)\chi(x)]$$

$$= \nabla_\mu\phi(x)\chi(x) + \phi(x)\nabla_\mu\chi(x) + a\nabla_\mu\phi(x)\nabla_\mu\chi(x)$$

Dondi & Nicolai, 1977

Problems for interacting theory

- SUSY algebra on elementary fields
- Not on polynomials of fields

- Result: *O(a)* artifact:

$$\delta_\epsilon S = i[\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}, S] = ia(\epsilon^\alpha X_\alpha - \bar{\epsilon}_{\dot{\alpha}} \bar{X}^{\dot{\alpha}})$$

- But we send $a \to 0$ at the end of our calculations, so who cares?

Sketch of the problem

- Consider a (global) supersymmetry Ward identity.
- In the continuum,

$$\langle [Q_\alpha, \mathcal{O}] \rangle = 0$$

- But on the lattice, where the action is not invariant,

$$\langle [Q_\alpha, \mathcal{O}] \rangle = \langle [Q_\alpha, S] \mathcal{O} \rangle = a \langle X_\alpha \mathcal{O} \rangle$$

- Obviously if

$$\langle X_\alpha \mathcal{O} \rangle \sim a^{-\ell}, \quad \ell \geq 1$$

  then we have a problem.

Fine-tuning

- We add counterterms to the bare action and then

$$QS = X_0 + aX_1 + \cdots$$

$$\langle QO \rangle = \langle X_0 O \rangle + a \langle X_1 O \rangle + \cdots$$

- By tuning the counterterms we can arrange for the necessary cancellations.

A more detailed examination

- In QCD we use local WI

$$\langle \partial_\mu A^a_\mu(x) P^b(y) \rangle = 2m_0 \langle P^a(x) P^b(y) \rangle + a \langle O^a_5(x) P^b(y) \rangle + \text{contact terms}$$

- Wilson term $\frac{a}{2}\overline{q} D^2 q \to O^a_5$ and $m_0 \to m_{cr}$

- Need renormalization for $a \to 0$ argument
- Remove power divergences

Testa, hep-th/9803147

Only need to tune bare mass

$$O_5^{Ra} = Z_5[O_5^a + \frac{1}{a}(Z_A - 1)\partial_\mu A_\mu^a + \frac{1}{a^2} 2Z'_P P^a] + \sum_j Z_5^{(j)} O_5^{(j)Ra}$$

$$\langle \partial_\mu A_\mu^{Ra}(x) P^{Rb}(y) \rangle = 2(m_0 - \frac{Z'_P}{a}) \frac{1}{Z_P} \langle P^{Ra}(x) P^{Rb}(y) \rangle + \frac{a}{Z_5} \langle O_5^{Ra}(x) P^{Rb}(y) \rangle$$
$$- a \sum_j \frac{Z_5^{(j)}}{Z_5} \langle O_5^{(j)Ra}(x) P^{Rb}(y) \rangle + \text{contact terms}$$

$$\to 2(m_0 - \frac{Z'_P}{a}) \frac{1}{Z_P} \langle P^{Ra}(x) P^{Rb}(y) \rangle + \text{contact terms}$$

$$A_\mu^{Ra} = Z_A A_\mu^a, \quad P^{Ra} = Z_P P^a$$

Slight generalization for N=1 SYM

- Failure of Leibnitz rule → O(a) term

$$\langle \partial_\mu S_\mu(x) O(y)\rangle = m_0 \langle \chi(x) O(y)\rangle + a\langle O_{11/2}(x) O(y)\rangle + \text{contact terms}$$

$$O^R_{11/2} = Z_{11/2}[O_{11/2} + \tfrac{1}{a}(Z_S - 1)\partial_\mu S_\mu + \tfrac{1}{a} Z_T \partial_\mu T_\mu + \tfrac{1}{a^2} Z_\chi \chi] + \textstyle\sum_j Z^{(j)}_{11/2} O^{(j)R}_{11/2}$$

$$S_\mu = -\sigma_{\rho\nu}\gamma_\mu \mathrm{Tr}(F_{\rho\nu}\lambda), \quad T_\mu = 2\gamma_\nu \mathrm{Tr}(F_{\mu\nu}\lambda), \quad \chi = \sigma_{\mu\nu}\mathrm{Tr}(F_{\mu\nu}\lambda)$$

$$\langle \partial_\mu S^R_\mu(x) O^R(y)\rangle = (m_0 - \tfrac{Z_\chi}{a})\langle \chi(x) O^R(y)\rangle + a \cdot \text{finite} + \text{contact terms}$$

$$S^R_\mu = Z_S S_\mu + Z_T T_\mu$$

- Z_T/Z_S must be determined to test for SUSY
- Fine-tuning just m_0 agrees w/ expectations

Farchioni et al. 2001

Wilson fermion N=4 SYM

- How bad is an entirely conventional approach, say using Wilson fermions?
- We cannot impose SU(4) R symmetry because it is chiral, and Wilson fermions violate chiral symmetry.
- However, we can impose SO(4) flavor symmetry with the fermions in 4's (vector) and the scalars in 6 (antisymmetric tensor).
- To determine the number of fine-tunings, we write down the most general renormalizable action consistent with these constraints.

$$S = \int d^4x \ \mathrm{Tr}\{-\tfrac{1}{2g_r^2}F_{\mu\nu}F_{\mu\nu} + \tfrac{i}{g_r^2}\overline{\lambda}_i\overline{\sigma}^\mu D_\mu\lambda_i + \tfrac{1}{g_r^2}D_\mu\phi_m D_\mu\phi_m + m_\phi^2\phi_m\phi_m$$

$$+m_\lambda(\lambda_i\lambda_i+\overline{\lambda}_i\overline{\lambda}_i)+\kappa_1\phi_m\phi_m\phi_n\phi_n+\kappa_2\phi_m\phi_n\phi_m\phi_n+y_1(\lambda_i[\phi_{ij},\lambda_j]+\overline{\lambda}_i[\phi_{ij},\overline{\lambda}_j])$$

$$+y_2\epsilon_{ijkl}(\lambda_i[\phi_{jk},\lambda_l]+\overline{\lambda}_i[\phi_{jk},\overline{\lambda}_l])\}$$

$$+\int d^4x \ \{\kappa_3(\mathrm{Tr}\phi_m\phi_m)^2+\kappa_4\mathrm{Tr}\phi_m\phi_n\mathrm{Tr}\phi_m\phi_n\}$$

- We achieved the first three coefficients by rescaling the fermion and scalar.
- We are left with 8 parameters to fine-tune: hopeless.

- Goal of modern formulations: reduce the number of fine-tunings.
- Method: lattice symmetries that restrict the long distance effective action.

Twisted N=4

- We form the twisted rotation group from an SO(4) subgroup of the flavor (R symmetry) group SU(4):

$$SO(4)' = \mathrm{diag}[SO(4)_E \times SO(4)_R]$$

$$\lambda^I_\alpha \rightarrow \Psi_{\alpha\beta}$$

- Then it is natural to expand on the five gamma matrices (a=1,…,5):

$$\Psi = \frac{1}{2}\eta + \psi_a\gamma_a + \frac{i}{2}\chi_{ab}[\gamma_a, \gamma_b]$$

- Given the 5d language of the fermions, it is also natural to package up the bosons in a 5d way:

$$\mathcal{A}_a = A_a + iB_a, \quad \overline{\mathcal{A}}_a = A_a - iB_a$$

Q invariant action

$$S = \frac{1}{2g^2}(Q\Lambda + S_{\text{closed}})$$

$$\Lambda = \int d^4x \ \text{Tr}(\chi_{mn}\mathcal{F}_{mn} + \eta[\overline{\mathcal{D}}_m, \mathcal{D}_m] - \frac{1}{2}\eta d)$$

$$S_{\text{closed}} = -\frac{1}{4}\int d^4x \ \text{Tr}\epsilon_{mnrpq}\chi_{pq}\overline{\mathcal{D}}_r\chi_{mn}$$

$$Q\mathcal{A}_m = \psi_m, \quad Q\psi_m = 0, \quad Q\overline{\mathcal{A}}_m = 0$$

$$Q\chi_{mn} = -\overline{\mathcal{F}}_{mn}, \quad Q\eta = d, \quad Qd = 0$$

- We can decompose the 5d fields into a 4d language.

$$\mathcal{A}_a \to \mathcal{A}_\mu \oplus \phi, \quad \mathcal{F}_{ab} \to \mathcal{F}_{\mu\nu} \oplus \mathcal{D}_\mu \phi$$

$$[\overline{\mathcal{D}}_a, \mathcal{D}_a] \to [\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] \oplus [\overline{\phi}, \phi]$$

$$\psi_a \to \psi_\mu \oplus \bar{\eta}, \quad \chi_{ab} \to \chi_{\mu\nu} \oplus \bar{\psi}_\mu$$

- Then the action becomes

$$\int d^4x \ \mathrm{Tr}\{-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \tfrac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 + \tfrac{1}{2}[\overline{\phi}, \phi]^2 + (\mathcal{D}_\mu\phi)^\dagger\mathcal{D}_\mu\phi - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]}$$

$$-\overline{\psi}_\mu\mathcal{D}_\mu\overline{\eta} - \overline{\psi}_\mu[\phi, \psi_\mu] - \eta\overline{\mathcal{D}}_\mu\psi_\mu - \eta[\overline{\psi}, \overline{\eta}] - {}^*\chi_{\mu\nu}\overline{\mathcal{D}}_\mu\overline{\psi}_\nu - {}^*\chi_{\mu\nu}[\overline{\phi}, \chi_{\mu\nu}]\}$$

$${}^*\chi_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\chi_{\rho\sigma}$$

- Twist of N=4 SYM Marcus made explicit (1995).

Lattice discretization

- In the lattice theory we switch to link variables for the gauge fields

$$\mathcal{U}_a(x) = e^{\mathcal{A}_a(x)}, \quad \overline{\mathcal{U}}_a(x) = \mathcal{U}_a^\dagger(x) = e^{-\overline{\mathcal{A}}_a(x)}$$

- Physically, $\mathcal{U}_a(x)$ is a link that goes from

$$x \to x + ae_a$$

and $\overline{\mathcal{U}}_a(x)$ is a link between the same pair of sites but going in the opposite direction.

Catterall, 0712.2532

- The five e_a are basis vectors of the A_4^* lattice.

$$
\begin{aligned}
e_1 &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}}\right) \\
e_2 &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}}\right) \\
e_3 &= \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}\right) \\
e_4 &= \left(0, 0, -\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{20}}\right) \\
e_5 &= \left(0, 0, 0, -\frac{4}{\sqrt{20}}\right)
\end{aligned}
$$

- These vectors are also used to construct an important orthogonal matrix

$$\mathcal{O}_{a\mu} = e_a^{\mu}, \quad \mathcal{O}_{a5} = \frac{1}{\sqrt{5}}, \quad a = 1, \ldots, 5, \quad \mu = 0, \ldots, 3$$

- The bosonic fields of the usual formulation of N=4 SYM are obtained as

$$\mathcal{V}_{\mu} = A_{\mu} + i\phi_{\mu+1} = \mathcal{O}_{a\mu}\mathcal{A}_a, \quad \phi_5 + i\phi_6 = \mathcal{O}_{a5}\mathcal{A}_a$$

Unsal, hep-th/0603046

- Under gauge transformations, the link variables transform in the usual way:

$$\mathcal{U}_a(x) \rightarrow g(x)\mathcal{U}_a(x)g^\dagger(x+e_a)$$
$$\overline{\mathcal{U}}_a(x) \rightarrow g(x+e_a)\overline{\mathcal{U}}_a(x)g^\dagger(x)$$

- The transformations of all of our other fields are dictated by this index related prescription

$$\eta(x) \rightarrow g(x)\eta(x)g^\dagger(x), \quad \psi_a(x) \rightarrow g(x)\psi_a(x)g^\dagger(x+e_a)$$

$$\chi_{ab}(x) \rightarrow g(x+e_a+e_b)\chi_{ab}(x)g^\dagger(x)$$

- Next we have to figure out how to discretize the covariant derivatives.
- For the field strength, the following has the right continuum limit:

$$\mathcal{F}_{ab}(x) = \mathcal{D}_a^{(+)}\mathcal{U}_b(x) = \mathcal{U}_a(x)\mathcal{U}_b(x+e_a) - \mathcal{U}_b(x)\mathcal{U}_a(x+e_b)$$

- We also introduce derivatives for term 2 of the action:

$$[\overline{\mathcal{D}}_a, \mathcal{D}_a] \rightarrow \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(x) = \mathcal{U}_a(x)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x-e_a)\mathcal{U}_a(x-e_a)$$

- It is easy to see this has the right continuum limit.

- The lattice version of Q transformations is a fairly straightforward transcription from the continuum:

$$Q\mathcal{U}_a = \psi_a, \quad Q\psi_a = 0, \quad Q\overline{\mathcal{U}}_a = 0$$

$$Q\chi_{ab}(x) = \overline{\mathcal{F}}_{ab}(x) \equiv \overline{\mathcal{U}}_b(x+e_a)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x+e_b)\overline{\mathcal{U}}_b(x)$$

$$Q\eta = d, \quad Qd = 0$$

- Then the Q exact action requires the replacements in the "gauge fermion"

$$\chi_{ab}\mathcal{F}_{ab}: \quad \mathcal{F}_{ab}(x) = \mathcal{D}_a^{(+)}\mathcal{U}_b(x) = \mathcal{U}_a(x)\mathcal{U}_b(x+e_a) - \mathcal{U}_b(x)\mathcal{U}_a(x+e_b)$$

$$\eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] \rightarrow \eta\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(x) = \eta(x)[\mathcal{U}_a(x)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x-e_a)\mathcal{U}_a(x-e_a)]$$

$$S_{\mathrm{Q-exact}} = \sum_x Q\mathrm{Tr}\{\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a - \frac{1}{2}\eta d\}$$

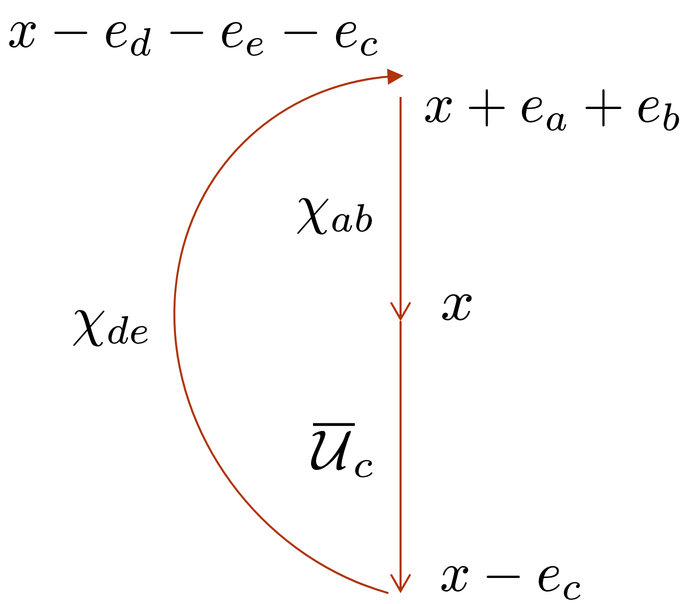
- The action has a shift invariance (using equations of motion):

$$\eta \rightarrow \eta + \epsilon\mathbf{1}$$

- The Q-closed term is a little more work

$$\chi_{de}\overline{\mathcal{D}}_c^{(-)}\chi_{ab} = \chi_{de}(\cdots)[\chi_{ab}(x)\overline{\mathcal{U}}_c(\cdots) - \overline{\mathcal{U}}_c(\cdots)\chi_{ab}(x-e_c)]$$

$$= \chi_{de}(x-e_d-e_e-e_c)[\chi_{ab}(x)\overline{\mathcal{U}}_c(x-e_c) - \overline{\mathcal{U}}_c(x-e_c+e_a+e_b)\chi_{ab}(x-e_c)]$$



- For the closure of this term, an important property is the lattice Bianchi identity

$$\epsilon_{abcde}\overline{\mathcal{D}}_c^{(-)}\overline{\mathcal{F}}_{ab} = 0$$

- If we have a renormalization scheme that preserves the lattice structure (including the symmetries), then we can enumerate the terms in the most general long distance effective action.
- There is only one Q-closed operator allowed by the lattice symmetries and it is already present.

$$S_{\mathrm{Q-closed}} = -\frac{\alpha_4}{4}\sum_x \epsilon_{abcde}\mathrm{Tr}(\chi_{de}\overline{\mathcal{D}}_c^{(-)}\chi_{ab})$$

- Q-exact terms must be fermionic, so they take the general form

$$Q\mathrm{Tr}[\Psi f(\mathcal{U},\overline{\mathcal{U}})]$$

- Taking into account the lattice gauge invariance and S_5 symmetry, we have (up to irrelevant operators)

$$Q\mathrm{Tr}(\chi_{ab}\mathcal{U}_a\mathcal{U}_b) - Q\mathrm{Tr}(\chi_{ab}\mathcal{U}_b\mathcal{U}_a) = Q\mathrm{Tr}(\chi_{ab}\mathcal{D}_a^{(+)}\mathcal{U}_b)$$

- With η we have lots of operators but shift invariance reduces to a few combinations

$$Q\mathrm{Tr}[\eta(x)\overline{\mathcal{U}}_a(x-e_a)\mathcal{U}_a(x-e_a)], \quad Q\mathrm{Tr}(\eta d)$$

$$Q\mathrm{Tr}[\eta(x)\mathcal{U}_a(x)\overline{\mathcal{U}}_a(x)], \quad Q\mathrm{Tr}\eta, \quad Q\{\mathrm{Tr}\eta\mathrm{Tr}(\mathcal{U}_a\overline{\mathcal{U}}_a)\}$$

$$\Downarrow$$

$$Q\mathrm{Tr}[\eta\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a], \quad Q\mathrm{Tr}(\eta d)$$

$$Q\mathrm{Tr}(\eta\mathcal{U}_a\overline{\mathcal{U}}_a) - \frac{1}{N}Q\{\mathrm{Tr}\eta\mathrm{Tr}(\mathcal{U}_a\overline{\mathcal{U}}_a)\}$$

- There is one other Q exact operator that we can write down

$$Q\mathrm{Tr}[\psi_a(x)\mathcal{U}_a(x+e_a)\overline{\mathcal{U}}_a(x+e_a)\overline{\mathcal{U}}_a(x)]$$

- In the continuum limit this becomes at leading order in the lattice spacing

$$Q\mathrm{Tr}(\psi_a\mathcal{A}_a) = \mathrm{Tr}(\psi_a\psi_a) = 0$$

due to the Grassmann nature of the field.

- Thus the renormalizable long distance theory is

$$S = \sum_x Q\mathrm{Tr}\{\alpha_1 \chi_{ab} \mathcal{D}_a^{(+)} \mathcal{U}_b + \alpha_2 \eta \overline{\mathcal{D}}_a^{(-)} \mathcal{U}_a - \frac{\alpha_3}{2} \eta d\}$$

$$+ \sum_x \beta_1 Q\{\mathrm{Tr}(\eta \mathcal{U}_a \overline{\mathcal{U}}_a) - \frac{1}{N} \mathrm{Tr}\eta \mathrm{Tr}(\mathcal{U}_a \overline{\mathcal{U}}_a)\}$$

$$- \frac{\alpha_4}{4} \sum_x \epsilon_{abcde} \mathrm{Tr}(\chi_{de} \overline{\mathcal{D}}_c^{(-)} \chi_{ab})$$

- Seems to be 4 fine-tunings. This is far fewer than a naive approach would yield.

- Act with Q and then rescale the fermions and auxiliary field:

$$\eta \to \lambda_\eta \eta, \quad \chi_{ab} \to \lambda_\chi \chi_{ab}, \quad \psi_a \to \lambda_\psi \psi_a, \quad d \to \lambda_d d$$

- The action becomes

$$\mathrm{Tr}\Big\{ -\alpha_1 \overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} - \alpha_1 \lambda_\chi \lambda_\psi \chi_{ab} \mathcal{D}^{(+)}_{[a}\psi_{b]} + \alpha_2 \lambda_d d \overline{\mathcal{D}}^{(-)}_a \mathcal{U}_a - \alpha_2 \lambda_\eta \lambda_\psi \eta \overline{\mathcal{D}}^{(-)}_a \psi_a$$

$$-\tfrac{\alpha_3}{2}\lambda_d^2 d^2 - \tfrac{\alpha_4}{4}\lambda_\chi^2 \epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}^{(-)}_c \chi_{ab}\Big\} + \beta\Big\{\lambda_d \mathrm{Tr}(d\mathcal{U}_a\overline{\mathcal{U}}_a) - \lambda_\eta\lambda_\psi \mathrm{Tr}(\eta\psi_a\overline{\mathcal{U}}_a)$$

$$-\tfrac{1}{N}\lambda_d \mathrm{Tr}d\,\mathrm{Tr}(\mathcal{U}_a\overline{\mathcal{U}}_a) + \tfrac{1}{N}\lambda_\eta\lambda_\psi \mathrm{Tr}\eta\,\mathrm{Tr}(\psi_a\overline{\mathcal{U}}_a)\Big\}$$

- Use freedom to set

$$\alpha_1 \lambda_\chi \lambda_\psi = \alpha_1, \quad \alpha_2 \lambda_d = \alpha_1, \quad \alpha_2 \lambda_\eta \lambda_\psi = \alpha_1, \quad \alpha_4 \lambda_\chi^2 = \alpha_1$$

- Solution:

$$\lambda_\eta = \sqrt{\frac{\alpha_1^3}{\alpha_4 \alpha_2^2}}, \quad \lambda_\chi = \frac{1}{\lambda_\psi} = \sqrt{\frac{\alpha_1}{\alpha_4}}, \quad \lambda_d = \frac{\alpha_1}{\alpha_2}$$

- Define

$$\alpha_3' = \alpha_3 \left(\frac{\alpha_1}{\alpha_2}\right)^2, \quad \beta' = \beta \frac{\alpha_1}{\alpha_2}$$

- Action is now

$$\mathrm{Tr}\Big\{-\alpha_1\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab}-\alpha_1\chi_{ab}\mathcal{D}^{(+)}_{[a}\psi_{b]}+\alpha_1 d\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a-\alpha_1\eta\overline{\mathcal{D}}^{(-)}_a\psi_a$$

$$-\frac{\alpha_3'}{2}d^2-\frac{\alpha_1}{4}\epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}^{(-)}_c\chi_{ab}\Big\}+\beta'\Big\{\mathrm{Tr}(d\mathcal{U}_a\overline{\mathcal{U}}_a)-\mathrm{Tr}(\eta\psi_a\overline{\mathcal{U}}_a)$$

$$-\frac{1}{N}\mathrm{Tr}d\,\mathrm{Tr}(\mathcal{U}_a\overline{\mathcal{U}}_a)+\frac{1}{N}\mathrm{Tr}\eta\,\mathrm{Tr}(\psi_a\overline{\mathcal{U}}_a)\Big\}$$

- Only 2 fine-tunings:

$$\alpha_3'\to\alpha_1,\quad \beta'\to 0$$

Cf. clover fermions, also 2 fine-tunings.

The other 15 SUSYs

- The supercharge also has the KD structure

$$\mathcal{Q} = Q + Q_a\gamma_a + \frac{i}{2}Q_{ab}[\gamma_a, \gamma_b]$$

- We can work out the other 15 SUSYs using discrete R invariances of the action (on-shell). For *a* fixed and *b,c,*etc. not equal to *a,*

$$R_a :$$

$$\eta \to 2\psi_a, \quad \psi_a \to \frac{1}{2}\eta, \quad \psi_b \to -\chi_{ab}$$

$$\chi_{ab} \to -\psi_b, \quad \chi_{bc} \to \frac{1}{2}\epsilon_{bcagh}\chi_{gh}$$

$$\mathcal{D}_a \to \mathcal{D}_a, \quad \overline{\mathcal{D}}_a \to \overline{\mathcal{D}}_a, \quad \mathcal{D}_b \to \overline{\mathcal{D}}_b, \quad \overline{\mathcal{D}}_b \to \mathcal{D}_b$$

- This leads to the five SUSYs

$$Q_a\mathcal{A}_b = \frac{1}{2}\delta_{ab}\eta, \quad Q_a\overline{\mathcal{A}}_b = -\chi_{ab}, \quad Q_a\psi_b = \frac{1}{2}\delta_{ab}d_a + (1-\delta_{ab})[\overline{\mathcal{D}}_a, \mathcal{D}_b]$$

$$Q_a\chi_{bc} = -\frac{1}{2}\epsilon_{abcde}[\mathcal{D}_d, \mathcal{D}_e], \quad Q_a\eta = 0, \quad Q_a d_a = 0$$

$$d_a = [\overline{\mathcal{D}}_a, \mathcal{D}_a] - \sum_{m\neq a}[\overline{\mathcal{D}}_m, \mathcal{D}_m]$$

- Then there are 10 other discrete R symmetries:

$$R_{ab}:$$

$$\eta \to 2\chi_{ab}, \quad \psi_a \to \psi_b, \quad \psi_b \to -\psi_a, \quad \psi_c \to \frac{1}{2}\epsilon_{cabgh}\chi_{gh}$$

$$\chi_{ab} \to -\frac{1}{2}\eta, \quad \chi_{ac} \to \chi_{bc}, \quad \chi_{bc} \to -\chi_{ac}, \quad \chi_{gh} \to -\epsilon_{ghabc}\psi_c$$

$$\mathcal{D}_{a,b} \to \overline{\mathcal{D}}_{a,b}, \quad \overline{\mathcal{D}}_{a,b} \to \mathcal{D}_{a,b}, \quad \mathcal{D}_c \to \mathcal{D}_c, \quad \overline{\mathcal{D}}_c \to \overline{\mathcal{D}}_c$$

Then one gets 10 more supercharges by applying these to Q:

$$Q_{ab}\mathcal{A}_c = \frac{1}{2}\epsilon_{abcgh}\chi_{gh}, \quad Q_{ab}\overline{\mathcal{A}}_c = \delta_{ac}\psi_b - \delta_{bc}\psi_a, \quad Q_{ab}\psi_c = \epsilon_{abcgh}\overline{\mathcal{F}}_{gh}$$

$$Q_{ab}\chi_{cd} = \frac{1}{4}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})d_{ab} + \delta_{ac}[\mathcal{D}_b, \overline{\mathcal{D}}_d] - \delta_{bc}[\mathcal{D}_a, \overline{\mathcal{D}}_d]$$

$$Q_{ab}\eta = 2\mathcal{F}_{ab}, \quad Q_{ab}d_{ab} = 0$$

$$d_{ab} = -[\overline{\mathcal{D}}_a, \mathcal{D}_a] - [\overline{\mathcal{D}}_a, \mathcal{D}_a] + \sum_{m\neq a,b}[\overline{\mathcal{D}}_m, \mathcal{D}_m]$$

The equation $Q_{ab}d_{ab} = 0$ requires the EOM.

R_a and renormalization

- Returning to

$$Q\mathrm{Tr}\Big\{\alpha_1\chi_{ab}\mathcal{F}_{ab} + \alpha_2\eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - \tfrac{\alpha_3}{2}\eta d\Big\}$$

$$-\tfrac{\alpha_4}{4}\epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c\chi_{ab} + \beta\{\cdots\}$$

- Eliminate auxiliary

$$\mathrm{Tr}\Big\{-\alpha_1\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \frac{\alpha_2^2}{2\alpha_3}[\overline{\mathcal{D}}_a, \mathcal{D}_a]^2 - \alpha_1\chi_{ab}\mathcal{D}_{[a}\psi_{b]}$$

$$-\alpha_2\eta\overline{\mathcal{D}}_a\psi_a - \tfrac{\alpha_4}{4}\epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c\chi_{ab} + \beta\{\cdots\}$$

- Apply R_a to this and demand invariance
- In bosonic sector terms are interchanged, requiring

$$\alpha_1 = \frac{\alpha_2^2}{\alpha_3}, \quad \beta = 0$$

- In fermionic sector terms are interchanged, requiring

$$\alpha_1 = \alpha_2 = \alpha_4, \quad \beta = 0$$

- Thus R_a invariance forces SUSY long distance theory.

- Recall

$$\mathcal{U}_a = e^{\mathcal{A}_a}, \quad \overline{\mathcal{U}}_a = e^{-\overline{\mathcal{A}}_a}$$

- Implies under R_a

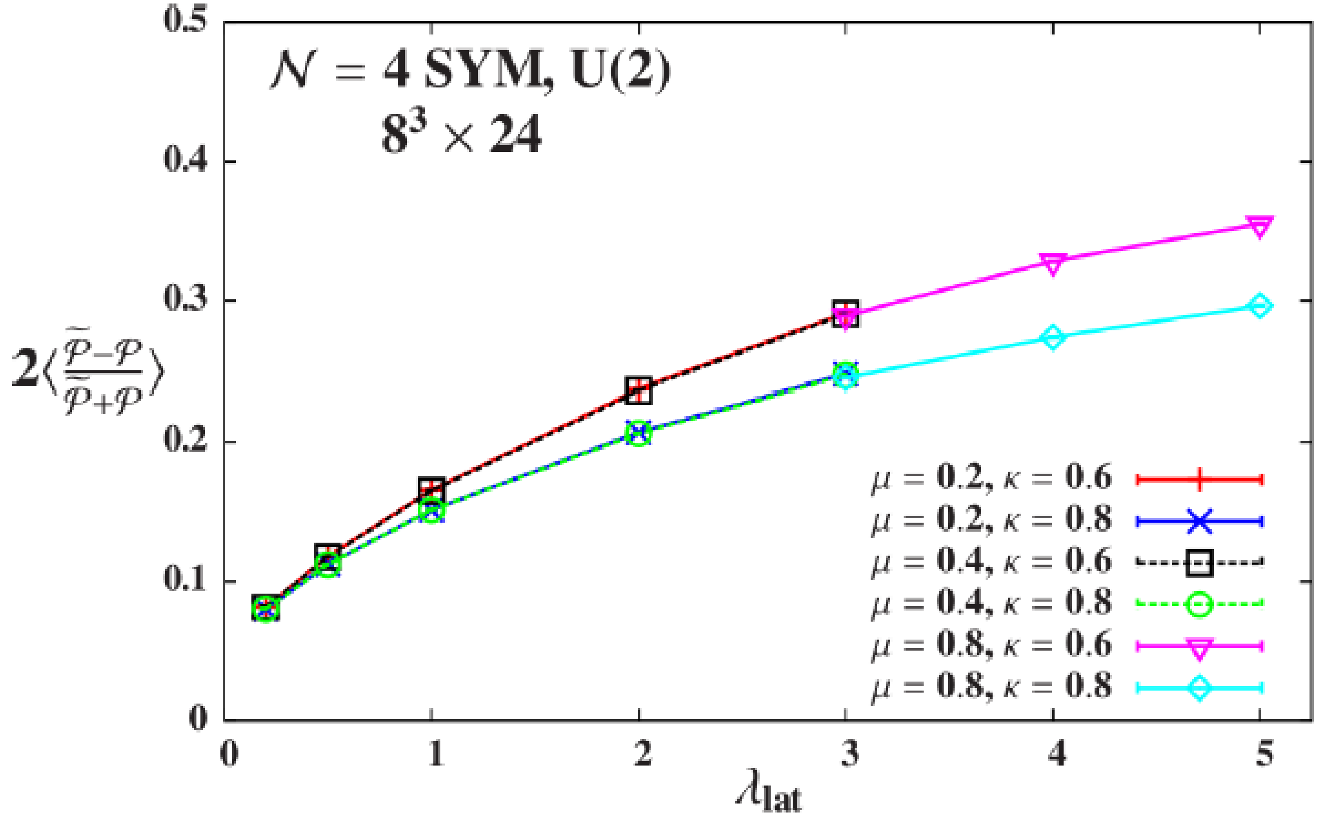
$$\mathcal{U}_a \rightarrow \mathcal{U}_a, \quad \overline{\mathcal{U}}_a \rightarrow \overline{\mathcal{U}}_a, \quad \mathcal{U}_b \rightarrow \overline{\mathcal{U}}_b^{-1}, \quad \overline{\mathcal{U}}_b \rightarrow \mathcal{U}_b^{-1}$$

- Thus a simple test of R_a restoration, and hence full N=4 SUSY restoration is

$$\langle \mathrm{Tr}\{\mathcal{U}_a(x)\mathcal{U}_b(x+e_a)\overline{\mathcal{U}}_a(x+e_b)\overline{\mathcal{U}}_b(x)\}\rangle$$

$$= \langle \mathrm{Tr}\{\mathcal{U}_a(x)\overline{\mathcal{U}}_b^{-1}(x+e_a)\overline{\mathcal{U}}_a(x+e_b)\mathcal{U}_b^{-1}(x)\}\rangle$$

- Amazing
- Due to exact symmetries of lattice theory



$\mathcal{N}$ = 4 SYM, U(2)
$8^3 \times 24$
$2\langle\frac{\tilde{\mathcal{P}}-\mathcal{P}}{\tilde{\mathcal{P}}+\mathcal{P}}\rangle$
$\lambda_{\rm lat}$
μ = 0.2, κ = 0.6
μ = 0.2, κ = 0.8
μ = 0.4, κ = 0.6
μ = 0.4, κ = 0.8
μ = 0.8, κ = 0.6
μ = 0.8, κ = 0.8
0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5

Blocking

- The arguments about the long distance effective action only hold if there is a real space renormalization group which preserves the lattice structure.
- This means that Q, S_5, gauge invariance and geometric interpretation of fields should survive the flow.
- Here we provide an explicit construction.

- The original lattice Λ may be described by

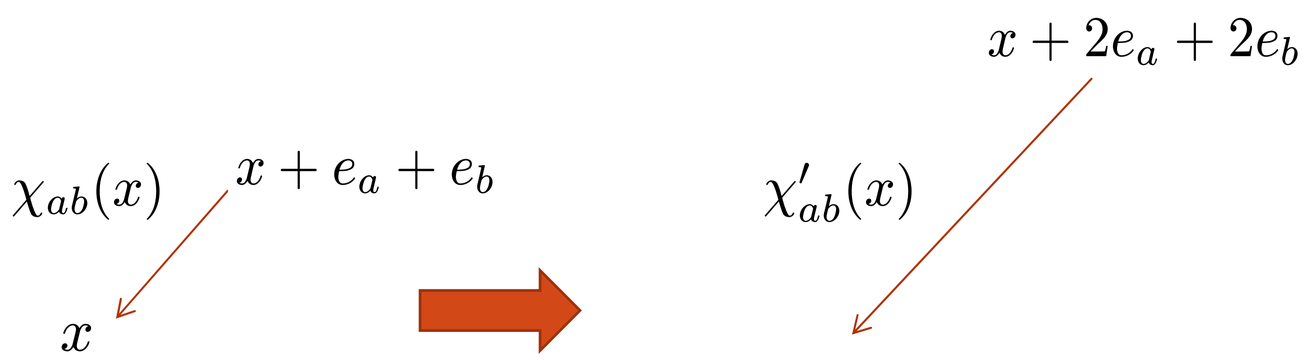
$$\Lambda = \{a \sum_{\mu=1}^{4} n_\mu e_\mu | n \in \mathbf{Z}^4\}$$

- where the e_μ are the first four of the five (degenerate) basis vectors of the A_4^* lattice described above.
- The blocked lattice will merely be doubled in every direction:

$$\Lambda' = \{2a \sum_{\mu=1}^{4} n_\mu e_\mu | n \in \mathbf{Z}^4\}$$

- From this point forward we will work in lattice units, setting $a = 1$

- The blocked fields will be denoted by primes.
- They must begin and end on sites of the blocked lattice Λ'.
- We want the geometric intepretation to survive the blocking.
- For example, $\chi'_{ab}(x)$ must begin on site $x + 2e_a + 2e_b$ and end on site x.



- One choice that achieves this is the following:

$$
\begin{aligned}
\mathcal{U}'_a(x) &= \mathcal{U}_a(x)\mathcal{U}_a(x+e_a), \quad \overline{\mathcal{U}}'_a(x) = \overline{\mathcal{U}}_a(x+e_a)\overline{\mathcal{U}}_a(x) \\
d'(x) &= d(x), \quad \eta'(x) = \eta(x) \\
\psi'_a(x) &= \psi_a(x)\mathcal{U}_a(x+e_a) + \mathcal{U}_a(x)\psi_a(x+e_a) \\
\chi'_{ab}(x) &= \frac{1}{2}[\overline{\mathcal{U}}_a(x+e_a+2e_b)\overline{\mathcal{U}}_b(x+e_a+e_b)\chi_{ab}(x) \\
&\quad +\overline{\mathcal{U}}_b(x+2e_a+e_b)\overline{\mathcal{U}}_a(x+e_a+e_b)\chi_{ab}(x)] \\
&\quad +[\overline{\mathcal{U}}_a(x+e_a+2e_b)\chi_{ab}(x+e_b)\overline{\mathcal{U}}_b(x) \\
&\quad +\overline{\mathcal{U}}_b(x+2e_a+e_b)\chi_{ab}(x+e_a)\overline{\mathcal{U}}_a(x)] \\
&\quad +\frac{1}{2}[\chi_{ab}(x+e_a+e_b)\overline{\mathcal{U}}_a(x+e_b)\overline{\mathcal{U}}_b(x) \\
&\quad +\chi_{ab}(x+e_a+e_b)\overline{\mathcal{U}}_b(x+e_a)\overline{\mathcal{U}}_a(x)]
\end{aligned}
$$

- This choice preserves the Q algebra, namely

$$Q\mathcal{U}'_a = \psi_a \mathcal{U}_a + \mathcal{U}_a \psi_a = \psi'_a$$

$$Q\psi'_a = -\psi_a \psi_a + \psi_a \psi_a = 0$$

$$Q\overline{\mathcal{U}}'_a = 0 \qquad Q\eta' = d = d' \qquad Qd' = 0$$

$$Q\chi'_{ab} = \overline{\mathcal{F}}'_{ab}$$

$$\overline{\mathcal{F}}'_{ab}(x) = \overline{\mathcal{U}}'_b(x + 2e_a)\overline{\mathcal{U}}'_a(x) - \overline{\mathcal{U}}'_a(x + 2e_b)\overline{\mathcal{U}}'_b(x)$$

- The last result, for $Q\chi'_{ab}$, is the only one that requires any significant computation.

Future directions

- Pfaffian phase
- RSRG calculations
- CTs, finite parts, two loops
- Other 15 SUSYs after RSRG, fine-tuning
- Strong coupling issues