Lattice models for anomalous field theories

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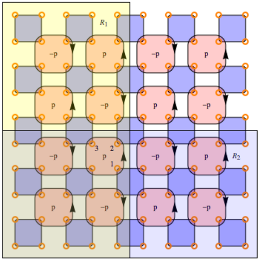
based on work with:
S. M. Kravec and Brian Swingle



Plan

Some interesting recent examples of cross-fertilization between condensed matter and high-energy theory:

1. Ideas about regularizing the Standard Model
   [Wen, You-BenTov-Xu]

2. Constraints on QFT from symmetry-protected topological (SPT) states
   [Shauna Kravec, JM, 1306.3992, PRL]

3. A machine for *explicitly* realizing SPT states
   [Shauna Kravec, JM, Brian Swingle, in progress]

Ideas from cond-mat are useful for high-energy theory

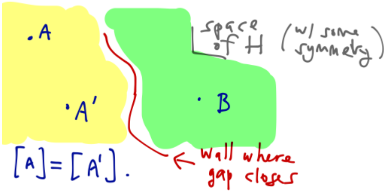
Goal: Identify obstructions
to symmetry-preserving regulators of QFT,
by thinking about certain states of matter in one higher dimension
which have an energy gap
(*i.e.* $E_1 - E_{gs} > 0$ in thermodynamic limit).

One outcome: [S. M. Kravec (UCSD), JM, arXiv:1306.3992, PRL]
Constraints on manifest electric-magnetic duality symmetry.

Realizations of symmetries in QFT and cond-mat

Basic Q: What are possible gapped phases of matter?

Def: Two gapped states are equivalent if they are adiabatically connected (varying the parameters in the **H** whose ground state they are to get from one to the other, without closing the energy gap).



One important distinguishing feature: how are the symmetries realized?

Landau distinction: characterize by *broken* symmetries

e.g. ferromagnet vs paramagnet, insulator vs SC. ✓

Mod out by Landau: "What are possible (gapped) phases that don't break symmetries?" How do we distinguish them?

One (fancy) answer [Wen]: topological order.

Basically, this means emergent, deconfined gauge theory.

Mod out by Wen, too

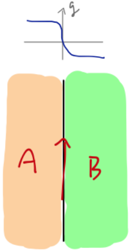
"What are possible (gapped) phases that don't break symmetries and don't have topological order?"

In the absence of topological order
('short-range entanglement' (SRE), hence simpler),
another answer: Put the model on the space with boundary.

A gapped state of matter in $d+1$ dimensions
with short-range entanglement
can be (at least partially) characterized (within some symmetry class of hamiltonians) by (properties of) its edge states
(*i.e.* what happens at an interface with the vacuum,
or with another such state).

SRE states are characterized by their edge states

Rough idea: just like varying the Hamiltonian in time to another phase requires closing the gap $\mathbf{H} = \mathbf{H}_1 + g(t)\mathbf{H}_2$, so does varying the Hamiltonian in space $\mathbf{H} = \mathbf{H}_1 + g(x)\mathbf{H}_2$.



- Important role of SRE assumption: Here we are assuming that the bulk state has short-ranged correlations, so that changes we might make at the surface cannot have effects deep in the bulk.

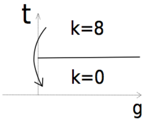
SPT states

Def: An *SPT state* (symmetry-protected topological state), protected by a symmetry group G is:
a SRE state, which is not adiabatically connected to a product state by local hamiltonians preserving G.

e.g.: free fermion topological insulators in 3+1d, protected by $U(1)$ and $\mathcal{T}$, have an odd number of Dirac cones on the surface.
One reason to care: if you gauge $H \subset G$, you get a state with topological order.

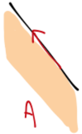
- Free fermion TIs classified [Kitaev: homotopy theory; Schneider et al: edge]

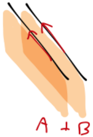
Interactions can affect the connectivity of the phase diagram in both directions:



- There are states which are adiabatically connected only via interacting Hamiltonians [Fidkowski-Kitaev, 0904.2197, Qi, Yao-Ryu, Wang-Senthil, You-BenTov-Xu].
- There are states whose existence requires interactions: *e.g.* Bosonic SPT states – w/o interactions, superfluid.

Group structure of SPT states

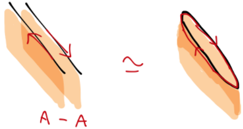




Simplifying feature:

SPT states (for given G) form a group:

-A : is the mirror image.



(Rueful comment: this cartoon can hide microscopic differences between $-A$ and the mirror image of A.)

Note: with topological order, even if we can gap out the edge states, there is still stuff going on (e.g. fractional charges) in the bulk. Not a group.

- [Chen-Gu-Wen, 1106.4772] conjecture: the group is $H^{D+1}(\mathrm{BG}, U(1))$.
- $\exists$ 'beyond-cohomology' states in $D \geq 3+1$ [Senthil-Vishwanath]
- The right group?: [Kitaev (unpublished), Kapustin, Thorgren].

Here: an implication of this group structure

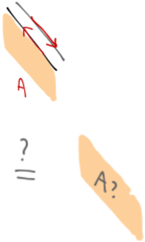
– which we can pursue by examples – is...

Surface-only models

Counterfactual:

Suppose the edge theory of an SPT state were realized *otherwise*

– intrinsically in D dimensions, with a local hamiltonian.



Then we could paint that the conjugate local theory on the surface without changing anything about the bulk state.

And then small deformations of the surface Hamiltonian, localized on the surface, consistent with symmetries, can pair up the edge states.

But this contradicts the claim that we could characterize the $D+1$-dimensional SPT state by its edge theory.

Conclusion: Edge theories of SPT_G states cannot be regularized intrinsically in D dims, *exactly* preserving on-site G –

"surface-only models" or "not edgeable".

[Wang-Senthil, 1302.6234 – general idea, concrete surprising examples of 2+1 surface-only states
Wen, 1303.1803 – attempt to characterize the underlying mathematical structure, classify *all* such obstructions
Metlitski-Kane-Fisher, 1302.6535; Burnell-Chen-Fidkowski-Vishwanath, 1302.7072]

Nielsen-Ninomiya result on fermion doubling

The most famous example of such an obstruction was articulated by Nielsen and Ninomiya:

It is not possible to regulate free fermions while preserving the chiral symmetry.

Recasting the NN result as a statement about SPT states

Consider free massive relativistic fermions in 4+1 dimensions (with conserved $U(1)$):

$$S = \int d^{4+1}x \bar{\Psi} \left(\partial\!\!\!/ + m \right) \Psi$$

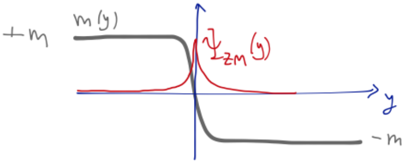
$\pm m$ label distinct Lorentz-invariant (P-broken) phases.

Domain wall between them hosts (exponentially-localized) 3+1 chiral fermions: [Jackiw-Rebbi, Callan-Harvey, Kaplan...]

One proof of this:

Couple to external gauge field $\Delta S = \int d^5 x A^\mu \bar{\Psi} \gamma_\mu \Psi$.

$$\log \int [D\Psi] e^{\mathbf{i} S_{4+1}[\Psi, A]} \propto \frac{m}{|m|} \int A \wedge F \wedge F$$



Galling fact: if we want the extra dimension to be finite, there's another domain wall with the antichiral fermions.

And if we put it too far away, the KK gauge bosons are too light...

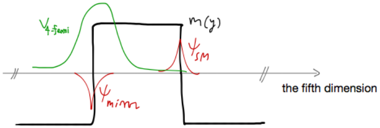
Loophole in NN theorem

But the SM gauge group is *not* anomalous, shouldn't need extra dimensions.

Loophole: Interactions between fermions!

Old idea: add four-fermion interactions (or couplings to other fields) which gap the mirror fermions, but not the SM, and preserve the SM gauge group G.

These interactions should explicitly break all anomalous symmetries.



This requires a right-handed neutrino. [Preskill-Eichten 1986, a lot of other work!]: SU(5) and SO(10) lattice GUTs.

Evidence for mirror-fermion mass generation via eucl. strong coupling expansion.

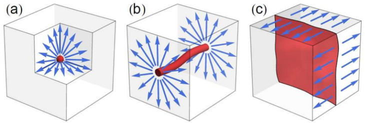
[Geidt-Chen-Poppitz]: numerical evidence for troubles of a related proposal in 1+1d.

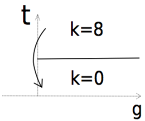
New evidence for a special role of $n_F = 16 \cdot n$

Collapse of free-fermion classification:
Dimensional recursion strategy [Wang-Senthil, Qi, Ryu-Yao, Wen]:

1. Consider neighboring phase where G is spontaneously broken $\langle \phi \rangle \neq 0$.
2. Proliferate defects of ϕ to reach paramagnetic phase.
3. Must ϕ-defects carry quantum numbers which make the paramagnet nontrivial?

Initial step: [Fidkowski-Kitaev]
edge of $8\times$ majorana chain is symmetrically gappable.
same refermionization as shows equivalence of GS and RNS superstrings, SO(8) triality.





[You-BenTov-Xu]: In 4+1d, with many G, the collapse again happens at $k = 8 \simeq 0$ ($\rightarrow$ 16 Weyl fermions per domain wall.)

Conclusion: This novel strategy for identifying obstructions to gapping the mirror fermions shows none when $n_F = 16n$.

2. An obstruction to a symmetric regulator

Strategy

Study a simple (unitary) gapped or topological field theory in 4+1 dimensions without topological order, with symmetry G.

Consider the model on the disk with some boundary conditions.

The resulting edge theory is
a "surface-only theory with respect to G"
– it cannot be regulated by a local $3+1$-dim'l model while preserving G.

This is the 4+1d analog of the "K-matrix approach" to 2+1d SPTs of [Lu-Vishwanath 12].

What does it mean to be a surface-only state?

These theories are perfectly consistent and unitary – they *can* be realized as the edge theory of some gapped bulk. They just can't be regularized in a local way consistent with the symmetries without the bulk.

1. It (probably) means these QFTs will not be found as low-energy EFTs of solids or in cold atom lattice simulations.
2. Why 'probably'? This perspective does not rule out emergent ("accidental") symmetries, not explicitly preserved in the UV.
3. It also does not rule out symmetric UV completions that include gravity, or decoupling limits of gravity/string theory.
   (UV completions of gravity have their own complications!)
   String theory strongly suggests the existence of Lorentz-invariant states of gravity with chiral fermions and lots of supersymmetry
   (the $E_8 \times E_8$ heterotic string, chiral matter on D-brane intersections, self-dual tensor fields...)
   some of which can be decoupled from gravity.

A simple topological field theory in 4+1 dimensions

Consider 2-forms B_{MN} in $4+1$ dimensions, with action

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R}\times\Sigma} B^I \wedge \mathrm{d}B^J$$

In $4\ell+1$ dims, K is a *skew*-symmetric integer $2N_B \times 2N_B$ matrix.

Note: $B \wedge \mathrm{d}B = \frac{1}{2}\mathrm{d}(B \wedge B)$.

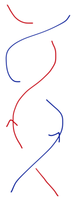
Independent of choice of metric on $\mathbb{R} \times \Sigma_{2p}$.

Related models studied in: [Horowitz 1989, Blau et al 1989, Witten 1998,

Maldacena-Moore-Seiberg 2001, Belov-Moore 2005, Hartnoll 2006]

[Horowitz-Srednicki]: coupling to string sources $\Delta S = \int_{\Gamma_I} B^I$

computes linking # of conjugate species of worldsheets Γ^I.



Simplest case ($N_B = 1$) is realized in IIB strings on $AdS_5 \times S^5$,

$B \equiv B_{NSNS}, C \equiv C_{RR}$:

$$S_{IIB} \ni \frac{1}{2\pi} \int_{AdS_5 \times S^5} F^{(5)}_{RR} \wedge B \wedge \mathrm{d}C = \frac{N}{2\pi} \int_{\mathbb{R}\times\Sigma} B \wedge \mathrm{d}C$$

Crucial hint: Type IIB S-duality acts by $B \leftrightarrow C$.

Strategy

1. Solve the model – when is it an EFT for an SPT state?
   Answer: when Pfaff$K = 1$.
2. Identify the edge states, and the symmetry G protecting them.
   (Whatever we get is surface-only with respect to G.)
   Answer: the edge theory is ordinary Maxwell theory, but with manifest electric-magnetic duality $(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E})$.

Comments:

1. Breaking Lorentz symm. is not enough to allow this symmetry: the edge theory we find is exactly the manifestly-duality-invariant model of [Schwarz-Sen 94].
2. **Corollary:** it is not possible to gauge electric-magnetic duality symmetry.
   ∃ recent literature with continuum arguments for this impossibility:
   [Deser 1012.5109, Bunster 1101.3927, Saa 1101.6064]
3. A similar construction in 6+1 dimensions produces a sector of the infamous $(2, 0)$ SCFT on the edge.

3. A solvable coupled-island construction of SPT states in $2+1$ dimensions

An SPT machine

A lot of effort has been put into classifying SPT states.

Fewer explicit constructions exist.

[EFT: Lu-Vishwanath; very plausible physical realization: Levin-Senthil]

Useful *e.g.* for understanding the phase transitions between them, and the topologically-ordered states that result upon gauging (subgroups of) G.

Virtues of our construction:

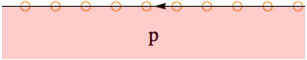
- Translation invariance not required. (Often translation invariance can protect an otherwise unprotected edge.)
- Uniform construction of domain wall operators.
  $\rightarrow$ [Levin-Gu] braiding statistics proof of nontriviality
- Illuminates connections between existing examples:
  [Levin-Gu ($\mathbb{Z}_2$), Chen-Liu-Wen ($\mathbb{Z}_2$), Chen-Gu-Liu-Wen (mysterious general formula?)]
- The 'duality' method of [Levin-Gu] was not available: gauging the bulk symmetry provides a (simpler?) construction of recent 'generalized string-net models' [Lin-Levin].

Non-onsite symmetry

An anomalous symmetry can be realized in a lattice model if it is *not on-site*: its action on one site depends on others.

This means you can't gauge it just by coupling to link variables (without coarse-graining first).

(Like chiral symmetry with staggered fermions.)



For example, at the edge of the Levin-Gu $\mathbb{Z}_2$ paramagnet,

$$\mathbf{S} = \prod_J \mathbf{X}_J \prod_J \mathbf{i}^{\frac{1}{2}(1-\mathbf{Z}_J\mathbf{Z}_{J+1})} = \prod_J \mathbf{X}_J \cdot \mathbf{i}^{\text{number of domain walls}}$$

A non-onsite symmetry $\mathbf{S}$ is nontrivial if $\mathbf{S} \neq \mathbf{U} \prod_j s_j \mathbf{U}^\dagger$ with $\mathbf{U}$ a *local symmetric unitary* (unitary evolution by a symmetric $\mathbf{H}$).

How to tell?? We will find a practical criterion below.

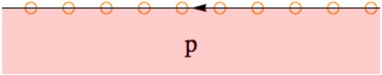
Focus on $\mathbb{Z}_N$ spins at each site:

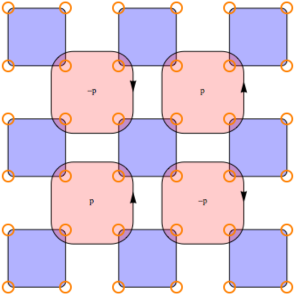
$$\mathbf{XZ} = \omega \mathbf{ZX}, \quad \omega \equiv e^{\frac{2\pi \mathbf{i}}{N}}.$$

$$\mathbf{Z}|n\rangle = \omega^n |n\rangle, \quad \mathbf{X}|n\rangle = |n-1\rangle, \quad n = 0..N-1 \quad (\text{mod } N)$$

An SPT machine

Given desired action of
non-onsite symmetry on edge:





Think of each bag as a site.

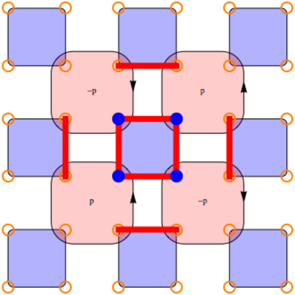
Couple together 'bags':

(Inspired by CZX model for $\mathbb{Z}_2$ [Chen-Liu-Wen, Swingle].)

$$\mathbf{H}_{\text{CZX}} = -\sum_{\square} \mathbf{b}_{\square} + h.c.$$

$$\mathbf{b}_{\square} \equiv \mathbf{XXXX}\mathcal{P}. \qquad [\mathbf{b}_{\square}, \mathbf{b}_{\square'}] = 0$$

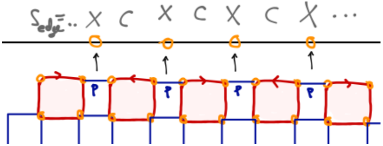
$$|\text{gs}\rangle = \prod_{\square} \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |nnnn\rangle$$



Symmetry-protected edge states

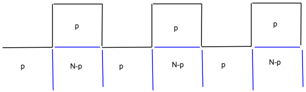
Note: **H** doesn't know about p; **S** does: ($[\mathbf{S}, \mathbf{H}_{\mathsf{CZX}}] = 0$)

$$\mathbf{S} = \prod_j \mathbf{X}_j \left(\prod_{\text{bags}} \prod_j C\left(\mathbf{Z}_j, \mathbf{Z}_{j+1}\right) \right)$$



Not onsite on edge: $\longrightarrow$

Rough edge realizes the desired **S** on edge modes:



Claim of robustness: perturbing $\mathbf{H}_{\mathsf{CZX}}$ by terms respecting **S**, you cannot remove this edge stuff.

i.e. no local, symmetric unitary can make $|gs\rangle_{\text{edge}}$ a product state. (Gapless or symmetry-breaking degeneracy.)

Shortcoming (?): requires bipartite graph of connections between bags.

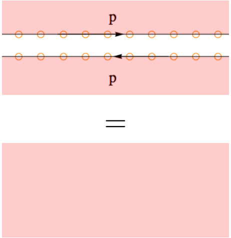
Note: I draw lattices for simplicity of drawing, but translation invariance is not at all necessary.

Bag link phases

$$\mathbf{S} = \prod_j \mathbf{X}_j \prod_{\text{bags}} \prod_j C\left(\mathbf{Z}_j, \mathbf{Z}_{j+1}\right)$$

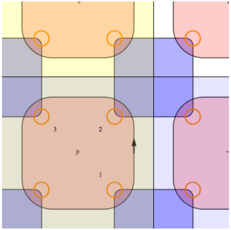
Wanted: A unitary operator $C(1,2) \equiv C(Z_1, Z_2)$ on two $\mathbb{Z}_N$-valued variables which satisfies the following three simple-looking conditions:

- group law : $\mathbf{S}^N = 1$



- gappability : $C_p(1,2)\,C_{-p}(1,2) = 1$

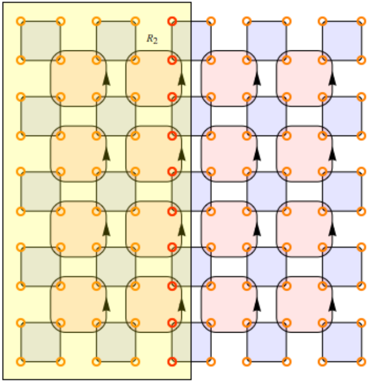
- non-triviality $\begin{pmatrix}\text{c-number DW} \\ \text{commutators}\end{pmatrix}$:



$$\frac{X_1 C_p(1,2) X_1^\dagger}{C_p(12)} \frac{X_3 C_{-p}(2,3) X_3^\dagger}{C_{-p}(23)} = \omega^p.$$

A uniform construction of domain-wall operators

$$W(R) \equiv \prod_{j\in R\setminus \text{last col}} \mathbf{X}_j \prod_{\text{bags in } R} \mathcal{P} \cdot \prod_{j\in bag} C(\mathbf{Z}_j, \mathbf{Z}_{j+1})$$



Acts like **S** in the interior of R.

Threads $2\pi/N$-flux along its boundary.

Becomes the string operator in the topologically-ordered model with G gauged and deconfined.

Same C!

Solution of conditions 1-3:

$$C(\mathbf{Z}_1, \mathbf{Z}_2) = e^{\frac{2\pi \mathbf{i} p}{N} \sum_{nm} \mathbf{P}_n(\mathbf{Z}_1)\mathbf{P}_m(\mathbf{Z}_2) a_{nm}}, \quad a_{nm} = n(1-m).$$

Eigenvalues are just Nth roots of unity.

Braiding of flux excitations

$\mathbf{W}_R$ generate symmetries:

$$[\mathbf{H}, \mathbf{W}_R] = 0 \ .$$

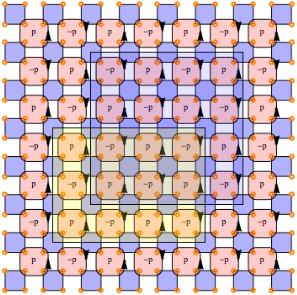
If domain walls intersect *once*:

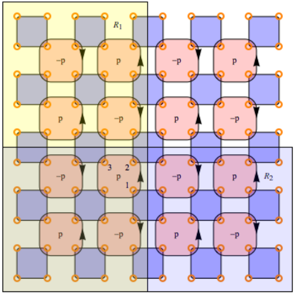
$$\mathbf{W}_{R_1}\mathbf{W}_{R_2} = \omega^p \mathbf{W}_{R_2}\mathbf{W}_{R_1}$$

must be represented on groundstates
(in fact, the whole spectrum)
$\implies$ nontrivial edge spectrum.

[Levin-Gu]

This gives a very practical condition for nontriviality of C.





Coarse-graining transformation

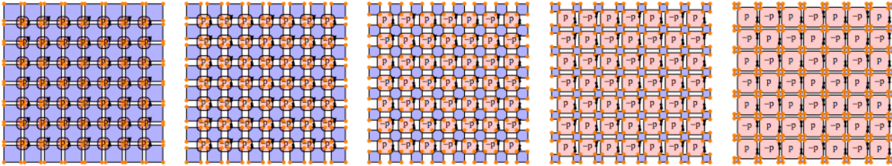
step 1: change basis to on-site symmetry:

$$\mathbf{U}\mathbf{S}_1\mathbf{U}^\dagger = \prod_j \mathbf{X}_j \qquad (\mathbf{U} \text{ known but, so far, ugly.})$$

Weirdness of **U**: it's a local unitary, but not continuously connected to $\mathbb{1}$ by local symmetric unitaries.

In this basis, easy to gauge. Alternative[Swingle]: diagonalize action on bags.

step 2: project to low-energy hilbert space = $\mathbb{Z}_N$ spins on sites J of bag graph:



$$\mathbf{H} = -\sum_J \mathbf{X}_J u_J(\mathbf{Z}) + h.c.$$

$$|\text{gs}\rangle = \mathbf{U} \otimes_J \left(\sum_n \frac{1}{\sqrt{N}} |n\rangle_J\right)$$

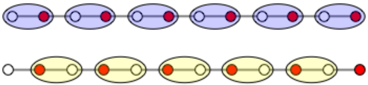
like [Levin-Gu] for $\mathbb{Z}_2$

should be interpreted in terms of fluctuating domain walls and junctions.

Non-onsite edge symmetry revisited

Again the action is by *duality*:

$\mathbf{S}_1$ exchanges the Jordan-Wigner (para)fermions for the spins with the JW (para)fermions for the disorder operators.



Symmetric edge Hamiltonian for $\mathbb{Z}_N$:

$$\mathbf{H} = -\sum_J \left(\mathbf{X}_J + \mathbf{Z}_{J-1}^p \mathbf{X}_J \mathbf{Z}_{J+1}^{-p}\right)$$

JW solution for $\mathbb{Z}_2$: Always in ferromagnetic (topological) phase.

For $\mathbb{Z}_N$: A simple lattice model for scalar with chiral $\mathbb{Z}_N$ symmetry.

Interesting generalizations

$G = \mathbb{Z}_N \times \mathbb{Z}_N$: $\exists$ a qualitatively different form of cocycle, related to a CS term of the form $a \wedge db$ rather than $a \wedge da$.

Coupled island construction:

introduce two sets of $\mathbb{Z}_N$ variables at each site of a CZX lattice, $\mathbf{X}, \mathbf{Z}, \tilde{\mathbf{X}}, \tilde{\mathbf{Z}}$.

$$\mathbb{Z}_N \times \mathbb{Z}_N : \quad S = \prod_j \mathbf{X}_j \prod_{bags} C(\tilde{\mathbf{Z}}), \quad \tilde{S} = \prod_j \tilde{\mathbf{X}}_j \prod_{bags} C(\mathbf{Z}).$$

$$\boxed{\mathbf{W}_{R_1}\tilde{\mathbf{W}}_{R_2} = \omega^p \mathbf{W}_{R_1}\tilde{\mathbf{W}}_{R_2}} \quad \left(\mathbf{W}_{R_1}\mathbf{W}_{R_2} = \mathbf{W}_{R_2}\mathbf{W}_{R_1}, \tilde{\mathbf{W}}_{R_1}\tilde{\mathbf{W}}_{R_2} = \tilde{\mathbf{W}}_{R_2}\tilde{\mathbf{W}}_{R_1}\right)$$

U(1): our original goal was to make a solvable model of the boson integer quantum Hall state [Levin-Senthil, Senthil-Regnault, Barkeshli].

A rotor at each site:

$$[\mathbf{n}, e^{\mathbf{i}\theta}] = e^{\mathbf{i}\theta}, \quad \mathbf{X}(\varphi)_j \equiv e^{\mathbf{i}\varphi \mathbf{n}_j}, \; \mathbf{X}(\varphi)\mathbf{Z}\mathbf{X}^\dagger(\varphi) = e^{\mathbf{i}\varphi}\mathbf{Z}.$$

$$\mathbf{b}_\square = \int_0^{2\pi} đ\theta \mathbf{X}(\theta)\mathbf{X}(\theta)\mathbf{X}(\theta)\mathbf{X}(\theta)\mathcal{P}$$

$$C_p(1,2) = e^{\mathbf{i}p \int_0^{2\pi} đ\theta_1 \int_0^{2\pi} đ\theta_2 \mathbf{P}_{\theta_1}(1)\mathbf{P}_{\theta_2}(2)(\theta_1\theta_2 + c_1\theta_1 + c_2\theta_2)}.$$

Questions

- For G = U(1), gauging the symmetry produces a model with $c_L \neq c_R$, but the model should still be solvable.
  Some tension with theorems of [Kitaev, honeycomb paper; Lin-Levin].
- We have not yet made precise the connection to group cohomology.
  The condition on the link phases that the DW commutator is a c-number should be the cocycle condition. Is it?
- Origin of bipartite restriction?!?
  In the continuum, there is no difference between $p \to -p$ and orientation reversal.
- Non-abelian G?
- 3d?

Concluding remark

Clearly the fruitful exchange of ideas between high energy theory and condensed matter theory continues.

The end.

Thanks for listening.