Expecting the Unexpected: Uniform Quantile Regression Bands, with an Application to Investor Sentiments

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Abstract
This paper develops uniform confidence bands for the linear quantile regression (QR) estimator in a time series setting. Such bands are useful for documenting the differences in responses at different quantiles of the conditional distribution. The inference procedure allows for serially correlated error terms, and is carried out through bootstrapping. I apply this method to the relationship between investor sentiments and future realized returns in quantiles, and show the following results: (1) there is pronounced heterogeneity in the slope parameter of the quantile regression; (2) this coefficient is slightly positive at lower quantiles while significantly negative at higher quantiles; (3) the negative relationship suggests that, at such quantiles, more optimistic predictions are correlated with lower future returns, a puzzling phenomenon that awaits further study.

Keywords: Investor Sentiments, Serially Correlated Errors, Uniform Quantile Regression Bands

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1 Introduction

Quantile regression (QR) analysis offers a natural and flexible framework for statistical analysis in the response model of conditional quantile functions. The appeal of quantile regression lies in a complete characterization of the entire conditional distribution, including the impact on distributional features beyond the mean. This paper explores the potential for quantile regression models as a tool for analyzing financial time series survey data.

This paper develops uniformly consistent confidence bands for the linear QR estimator. Compared to traditional point-wise confidence bands at each quantile, uniform confidence bands are consistent over quantiles. Such bands are useful for documenting the differences in responses at different quantiles of the conditional distribution. Conditional on the returns in the stock market, quantile estimates allows the slope coefficients to vary so that investor expectations provide unexpected results.

The contribution of this paper is to develop uniform quantile regression bands in a time series setting with serially correlated errors. I derive a procedure for inference, and then apply this method to weekly data on investor sentiments for S&P500 stock returns. In particular, I estimate the uniform confidence bands for the QR estimator of investor sentiments on the realized stock market returns. It is the first QR study in the relationship between returns and expectations at different quantiles of the stock distribution.

The empirical application shows that there is pronounced heterogeneity in the slope parameter of investor sentiments on future realized returns. This coefficient is slightly positive at lower quantiles while significantly negative at higher quantiles. The negative relationship suggests that, more optimistic predictions are correlated with lower future returns, a puzzling phenomenon that awaits further study.

The classical linear quantile regression model was first systematically proposed by Koenker and Bassett (1978). Koenker (2005) and its references offer a comprehensive analysis of the classical quantile regression model and its applications. Recent contributions to conditional quantile regression estimation includes Qu (2008), who works on testing for structural
changes in regression quantiles and derives the limiting distributions under the null hypoth-
esis. Also, Hardle and Song (2010) establish a strong uniform consistency rate for the QR
confidence bands with \textit{i.i.d} assumptions. Qu and Yoon (2015) derive inference theory for
constructing uniform confidence bands in a nonparametric setting by applying local linear
regressions to a grid of quantile rearrangements based on the \textit{i.i.d}. sample. Most existing QR
studies focus solely on \textit{i.i.d}. samples, while this paper allows for serially correlated errors.

From a methodological perspective, this paper is related to Wu and Zhou (2014), who con-
sider quantile structural change testing for linear models with a wide class of non-stationary
regressors and errors. From a technical perspective, this paper is closely related to Zhou
(2013), who first proposes a simple and unified bootstrap testing procedure which provides
consistent testing results under general forms of smooth and abrupt changes in the temporal
dynamics of the time series. From an empirical perspective, this paper continues the study
of Greenwood and Shleifer (2014), who compare survey data on expectations with expected
returns, showing evidence that is inconsistent with rational expectations representative in-
vestor models of returns. Based on the consistency of six survey sources of expectations with
the realized stock market returns, they showed that survey measures reflect the true beliefs
of many investors about future returns. This paper compares investor sentiments in different
quantiles of real stock returns, a further investigation into (ir)rational investment behavior.

The rest of the paper is organized as follows. Section 2 introduces the notation in a time
series setting and the assumptions in the baseline quantile regression model. The main goal
is to provide uniform confidence bands for the QR estimator. Section 3 focuses on the case
of serially correlated error terms and proposes a bootstrapping procedure. In section 4, I
empirically apply my method to data on U.S. investor sentiments in the S &P 500 market.
Section 5 provides robustness checks and section 6 concludes.
2 Quantile Regression Model

In this section, I briefly review the methodology of quantile regression. Here the issue of interest is the uniform confidence band for the quantile regression estimator. The advantage of the uniform confidence bands over quantiles is to make quantile estimators comparable at different quantiles.

2.1 Set up

Consider the parametric time series quantile regression model shown as follows.

\[ Q_{y|x_t}(\tau) = \beta(\tau)'x_t, \tau \in (0,1) \]

Let \( \{y_t\}_{t=1,2,...,T} \) denote the response time series for a sample size of \( T \) and \( \{x_t\} \) is a \( d \times 1 \) vector of random variables. Assume that for each given \( \tau \), the conditional quantile function of \( y_t \) given \( x_t \) is linear in \( x_t \). The research goal is to prove that the quantile regression estimator \( \hat{\beta}(\tau) \) is uniformly consistent over \( \tau \in [\omega, 1 - \omega], 0 < \omega < 1/2 \).

Denote the conditional distribution function \( F_t(y,x_t) = P(y_t \leq y|x_t) \). Conditional quantile functions of \( y_t \) given \( x_t \), denoted as \( Q_{y|x_t}(\tau) \), is obtained from solving \( F_t(y,x_t) = \tau \). Assuming integrability and uniqueness of the minimization solution, the quantile regression estimators \( \hat{\beta}(\tau) \) can be estimated by

\[
\hat{\beta}(\tau) = \arg \min_{\beta(\tau) \in \mathbb{R}^d} \sum_{t=1}^{T} \rho_\tau(y_t - \beta(\tau)'x_t),
\]

where the checking function \( \rho_\tau(u) = u \times \phi_\tau(u) = u \times (\tau - 1(u \leq 0)) \) and \( 1(.) \) is the indicator function. Denote the full sample subgradient \( S_T(\tau, \beta(\tau)) \) as

\[
S_T(\tau, \beta(\tau)) = T^{-1/2} \sum_{t=1}^{T} x_t \phi_\tau(y_t - \beta(\tau)'x_t)
\]

2.2 Assumptions

To obtain the limiting distribution, the following assumptions are imposed in the quantile regression model.
A1. \((y_t, x_t)_{t=1,...,T}\) is a strictly stationary and ergodic sequence such that \(T^{-1/2} \max_{1 \leq t \leq T} \|x_t\| = o_p(1)\) and \(E[\|x_t\|^2] < \infty\), where \(\|\cdot\|\) is the Euclidean norm.

A2. Let \(F(\cdot|x_t) = F_t(\cdot)\) denote the conditional distribution function of \(y_t\) given \(x_t\). \(F_t(\cdot)\) has continuous Lebesgue density \(f_t(\cdot) = f_t(\cdot|x_t)\) uniformly bounded away from 0 and \(\infty\) over quantiles \(\tau\). For a constant \(C > 0\), \(|f_t(y_1) - f_t(y_2)| \leq C|y_1 - y_2|\).

A3. \(E[\phi_{\tau}(y_t - \beta_0(\tau)'x_t)|x_t] = 0\), a.s. for some unique \(\beta_0(\tau) \in B \subset \mathbb{R}^p\), where \(\phi_{\tau}(u) = 1(u < 0) - \tau\) and \(\beta_0(\tau)\) is an interior point of the compact set \(B\) for each \(\tau\).

A4. There exist a random variable \(A_t\) and a constant \(0 \leq k_1 < 1/2\) such that \(T^{-1} \sum_{t=1}^T \|x_t\| \leq T^{k_1} \times A_t\). In addition, \(\sup E(A_t^{k_2}) < \infty\) for some \(k_2 > 2\).

A5. There exist \(k_3 \geq k_4 > 1\), and \(M < \infty, V < \infty\) such that for all \(T > 1, T^{-1} \sum_{t=1}^T [E(x_t'x_t)]^{k_3} \leq M, E[T^{-1} \sum_{t=1}^T (x_t'x_t)^{k_4}] \leq V, (k_3 - 1)/(k_4 - 1) > 1 + 2k_1\). If \(E(x_t'x_t)^2 \leq W < \infty \forall t\), we can take \(k_3 = 2, k_4 = 3/2\).

A6. \(T^{-1} \sum_{t=1}^T x_t x_t' = J_0 + o_p(1)\), where \(J_0\) is a finite, symmetric and positive definite matrix. \(T^{-1} \sum_{t=1}^T f_t(\beta_0(\tau)'x_t)x_t x_t' = H_0(\tau) + o_p(1)\) holds uniformly over \(\tau\) where \(H_0(\tau)\) is a \(d \times d\) symmetric and positive definite matrix for each \(\tau\).

A1 brings in the stationary time series setting. A2 is standard in quantile regression literature, allowing heteroskedasticity. It also imposes that the conditional densities are smooth in some neighborhood of \(F_t^{-1}(\tau)\) uniformly over time and quantile. A3 rules out the misspecification in the quantile regression model and provides an unique identification condition for the QR estimator \(\beta_0(\tau)\).

A4 imposes some restrictions on the heteroskedasticity. It also rules out trending regressors, under which the limiting distribution of the test statistics will be different. The assumption involves the behavior of the conditional quantile function in some neighborhood of the specific quantile. A5 is used to ensure the tightness of certain sequential weighted...
empirical process. The stochastic equicontinuity of the sequential empirical process is based on estimated quantile regression residuals, which is needed to establish weak convergence of $S_T(\tau, \beta)$. A6 provides general assumptions fairly to facilitate the derivation of the asymptotic results.

Regarding the error terms, I introduce some notation here. For a random variable $X$, let $||X||_q := (E|X|^q)^{1/q}$ be its $L_q$ norm. Denote filtrations $\varphi_i = (..., \eta_{i-1}, \eta_i)$ and $\varphi_i^{(j)} = (..., \eta_{j-1}, \eta_j', \eta_{j+1}, ..., \eta_i)$, $(j \leq i)$, where $\{\eta_j\}_j=-\infty, \{\eta_j'\}_j=-\infty$ are i.i.d. random variables.

Denote the error term in the quantile regression model by $e_t(\tau) = y_t - \beta(\tau)'x_t$. The error term could also be expressed as $e_t(\tau) = G_\tau(i, \varphi_{t,l})$. The filtration $\varphi_{t,l} = \{..., \epsilon_{0,l}, \epsilon_{1,l}, ..., \epsilon_{t,l}\}$ satisfies that for $l \neq s$, $\{\epsilon_l\}$ and $\{\epsilon_s\}$ are independent i.i.d. random variables.

**Proposition 1** Under Assumption 1-5 and for large $T$,

$$\sqrt{T}(\hat{\beta}(\tau) - \beta_0(\tau))H_0 - [\hat{S}_T(\tau, \hat{\beta}(\tau)) - S_T(\tau, \beta_0(\tau))] = o_p(1),$$

uniformly in $\tau$. $\hat{\beta}(\tau)$ is the quantile regression estimator of $\beta_0(\tau)$.

$$H_0 = \lim_{T \to \infty} T^{-1} \sum_{t=1}^{T} f_t(F_t^{-1}(\tau))x_t'x_t.$$  \hspace{1cm} (4)

Proposition 1 doesn’t depend on whether the error terms in the quantile regression are i.i.d. or serially correlated. This is the first step towards weak convergence of the QR estimator in the next section. The proof proceeds by establishing the uniform consistency of the quantile regression estimators over the quantiles. A detailed proof is shown in the technical Appendix.

### 3 Serially Correlated Errors

In this section, I derive the uniformly consistent confidence bands for QR estimators when the error terms, $e_t(\tau) = y_t - \beta(\tau)'x_t$, are serially correlated. Based on Wu and Zhou (2014), I apply the bootstrapping procedure, which is consistent for time series quantile regression procedures with both abruptly and smoothly time-varying temporal dynamics. The difference is that in this paper, serial correlation is considered in the error terms only, given that the error term is independent of $\{x_t\}_{t=1}^T$.  


Recall the error term could also be expressed as \( e_t(\tau) = G_{\tau}(i, \varphi_{t,l}) \). Serial correlation of the error term is embedded in the filtration \( \varphi_{t,l} \). A broad class of classic time series, such as invertible ARMA process, can be calculated through this measure. The following assumption S1 focuses on the dependence measure and smoothness of the error process \( e_t(\tau) \), given that the QR estimator \( \hat{\beta}(\tau) \) is consistent.

**S1.** Assume the error term \( e_t(\tau) = G_{\tau}(i, \varphi_{t,l}) \) satisfies that \( \forall i, j \in (0, 1), \| (G_{\tau}(i, \varphi_{t,l}) - G_{\tau}(j, \varphi_{t,l})) / |i - j| \|_v \leq C \) for some constant \( v > 1 \). The dependence measure of the error term in \( L_v \) norm satisfies \( \| G_{\tau}(i, \varphi_{t,l}) - G_{\tau}(i, \varphi_{t,\ast}) \|_v \leq M \chi \) where \( M \) is sufficient large constant and \( \chi \in (0, 1) \).

Assumption S1 requires that the process \( e_t \) to be short range dependent with exponentially decaying dependence measures. It covers a broad class of serially correlated error terms, such as invertible ARMA process. Note that serial correlation does not affect the unbiasedness or consistency of QR estimators, but it does affect their efficiency.

Recall that in the quantile checking function, we have \( \phi_{\tau}(u) = \tau - 1(u \leq 0) \), which is the left derivative of \( \rho_{\tau}(u) = \tau u^+ + (1 - \tau)(-u)^+ \) and \( 1(.) \) is the indicator function. Under assumption S1, based on Wu and Zhou (2014), on a richer probability space, there exists a zero-mean Gaussian process \( U_t(\tau) \), with covariance function \( \gamma(t, s) = \int_0^{\min(t,s)} \Sigma^2_{\tau}(r) dr \), s.t.

\[
|S_T(\tau, e_t) - U(\tau)| = o_p(T^{-1/4} \log^2 T). \tag{5}
\]

where \( S_T(\tau, e_t) = T^{-1/2} \sum_{t=1}^T x_t \phi_\tau(e_t) \) and \( \Sigma^2_{\tau}(t) = \sum_{\nu} Cov(\phi_\tau(e_t(\tau))x_t, \phi_\tau(e_{\nu}(\tau))x_{\nu}) \). The above equation shows that \( U(\tau) \) could be the limiting approximation process of \( S_T(\tau, e_t) \) under serially correlated error terms.

The inference under serially correlated errors suggests to simulate the data-driven Gaussian process \( U(\tau) \) in equation (5). Compared to the algorithm and simulation procedure in the previous section, the covariance structure of \( U(\tau) \) is obtained through gradient-based process bootstrapping according to Wu and Zhou (2014). I first introduce the following theorem regarding the limiting distribution of QR estimators and then propose the simulation procedure based on the following theorem.
Theorem 1 Suppose assumptions A1-5 and S1 holds, we have the following weak convergence in distribution:

\[ \sqrt{T}|\hat{\beta}_T(\tau) - \beta_0(\tau)| \Rightarrow_{\tau \to \infty} |H_0^{-1}(\tau)U(\tau)| \]

where \( U(\tau) \) is a \( p \)-dimensional zero-mean Gaussian process defined in Equation (8).

The proof of Theorem 1 follows Proposition 1 and property of gradient vectors \( S_T(\tau, \beta(\tau)) \). Here I apply a bootstrap process to estimate \( U(\tau) \). Define

\[ \hat{H}_c(t, \tau) = \sum_{t=1}^{T} \frac{\phi(\hat{e}_t(\tau)/c_t)x_t x'_t}{Tc_t} \]

\( \hat{H}_c(t, \tau) \) is an extension of the Powell’s Sandwich and can be viewed as a progressive local constant kernel estimation of integrated conditional density. \( c_t \) is the bandwidth chosen based on the minimum volatility method. It is also a uniformly consistent estimator of \( H_0(\tau) \) under serial correlation.

Based on Theorem 3.3 in Wu and Zhou (2014), under condition A1-5 and S1, as \( c_t \to 0, Tc_t^3 \to \infty \), we have the following equation holds uniformly over \( \tau \):

\[ |\hat{H}_c(\tau) - H_0(\tau)| = O_p(n^{-1/2}c_t^{-1/2} + \log^{10} T/(Tc_t^3) + c_t^2) \]

On the other hand, the gradient-based process \( \Psi_m(t) \) can be used as a uniformly consistent estimator of \( U_\tau(t) \). Given the sample size \( T \), we define

\[ \Psi_m = \sum_{t=1}^{T} (m(T-m+1))^{-1/2}(\hat{\omega}_{t,m} - m^{-1} \hat{\omega}_{1,\tau})R_t, \]

where \( \omega_{j,m} = \sum_{r=j}^{j+m-1} \phi_r(\hat{e}_rT(\tau))x_r \) and \( \{R_t\}_{t=1}^{T} \) are i.i.d. standard normals which are independent of \( \{\hat{\omega}_i\}_{i=-\infty}^{\infty} \). \( \hat{\Lambda}_c(t, \tau) \) converges to \( H_0(\tau) \) in probability and \( \hat{\Psi}_{m,n}(t) \) are consistent estimate of \( U(\tau) \) with the uniform topology.

The following corollary facilitate the construction of the confidence bands.

Corollary 2 Under Assumption 1-6, the uniform confidence bands \( \hat{C}_\alpha(\tau) \) for the QR estimator \( \beta(\tau) \) satisfy that \( \lim_{T \to \infty} P(\hat{\beta}(\tau) \in \hat{C}_\alpha(\tau), \forall \tau \in [\omega, 1-\omega]) = \alpha \),

\[ \hat{C}_\alpha(\tau) = [\hat{\beta}(\tau) : |\sqrt{T}(\hat{\beta}(\tau) - \beta_0(\tau))/\hat{\Sigma}(\tau)| \leq U_\alpha], \]

where \( \hat{\Sigma}(\tau) \) is the estimated uniform standard deviation of the QR estimator and \( U_\alpha \) is the \( (1-\alpha) \) percentile of \( \sup_{\tau}|U_\tau| \).
When the QR estimator \( \hat{\beta}(\tau) \) is a scalar, the uniform quantile regression bands can be expressed explicitly as follows \( \hat{C}_\alpha(\tau) = [\hat{\beta}(\tau) - \hat{\Sigma}(\tau)U_\alpha, \hat{\beta}(\tau) + \hat{\Sigma}(\tau)U_\alpha] \). \( U_\alpha \), as the \( (1 - \alpha) \) percentile of \( \sup_\tau |U_\tau| \), could be obtained through simulations.

The uniform confidence bands under serially correlated errors change in the derivation of the critical value. The simulation procedure is as follows: 1) select \( m \) and \( c_t \) based on minimum volatility (MV) method in section 4 of Zhou (2003), first advocated in Politis et al (1999); 2) generate \( B \) times (e.g. \( B=10000 \}) \{\Psi_m\} \) for sample size \( T \), and get \( \hat{H}_{c_t}(i,\tau) \), \( i = 1, ..., T \); 3) calculate \( E_b(\tau) = \hat{H}_{c_t}^{-1}(T,\tau)\Psi_m(\tau) \) for every given \( \tau \in [0.05, 0.95] \); 4) let \( E_b = \sup_\tau E_b(\tau) \). Let \( E(1) \leq E(2) \leq ... \leq E(B) \) be the order statistics of \( E_b \). Then \( E(\lceil(1-\alpha)B\rceil) \) is the level \( \alpha \) critical value for the QR estimator coefficient uniform confidence bands.

Bandwidth \( c_t \) is selected based on Zhou (2013). First, choose suitable interval \( I \) and divide equally into 99 pieces so that we have 100 points over \( I \). Optimal \( c_t \) lies inside the interval. Second, for each point value \( h_i \) in the interval \( I \), use it as bandwidth to calculate \( \hat{H}_{h_i}(1,\tau) \). Let \( C(i) \) be the maximal value of the right hand side of equation (7). Use \( T^{-1/2} \sum i \phi(\hat{e}_i(\tau)) \) and \( \hat{H}_{h_i} \) respectively. Define \( D(i) = \frac{1}{2k} \sum j=i-k[C(j) - \frac{1}{2k+1} \sum j=i-k C(i)]^2 \) and the minimum of \( D(i) \) is the optimal bandwidth selection for \( c_t \) in the simulation.

The choice of \( m \) satisfies that \( m = m(T) \to \infty \), and \( m(T)\log^7 T/\sqrt{T} \to 0 \). Assume \( m(T) \) is the order of \( T^{1/3} \) except a factor of multiplicative logarithm and \( c_t \) is the order of \( T^{-1/3} \) except a factor of multiplicative logarithm. Since \( \sup_{\tau \in (\omega,1-\omega)} |\hat{H}_{c_t}(\tau) - H(\tau)| \to 0 \) in probability and \( \Psi_{m,T}(t) \Rightarrow U(\tau) \) on \( C(0,1) \) with the uniform topology, the bootstrap implementation procedure converges to the optimal \( 1/\sqrt{T} \) parametric rate.

### 3.1 Simulation Studies

I perform a simulation study on the finite sample coverage probabilities of the uniform confidence bands under \( i.i.d. \) and serially correlated error terms. We are interested in investigating whether serially correlated errors would result in inaccurate coverage probabilities. For this purpose, consider the simplest linear quantile regression model \( y_t = 1 + x_t + u_t, t = 1, ..., T \).
The data generating process follows that $x_t$ are $\chi^2(3)/3$. I calculate the simulated coverage probabilities under two different error structures. Case I simulates the left hand side by drawing $u_t$ i.i.d. Uniform(0,1) and bootstrap $B=10000$ for sample size $n=250$ and 500. Case II uses the stationary AR(1) process: $u_t = 0.5u_{t-1} + \epsilon_t$ and $\epsilon_t$ are i.i.d. $N(0, 1)$.

Table 1 lists the simulated coverage probabilities under the two cases. We can see as the sample size increases, the simulated coverage rates increases. The simulation shows that the method of deriving uniform confidence bands for the quantile regression estimator works well under the nominal level of 5% and 10%.

4 Empirical Application to Investor Sentiments

As quantile regression estimates the conditional quantile functions by minimizing the quantile absolute deviations under the asymmetric loss function, there are a few advantages of applying this methodology to the financial time series analysis, for example the empirical application in this section.

Quantile regression model estimates the effect of X without making assumptions about error terms. The unknown features of stock market returns and investor expectations are still under research. So are the true relationship between investors expectation and realized stock market returns. Therefore, it is worthwhile to develop the quantile regression methodology for the financial time series.

As more evidence of asymmetric loss is revealed in the stock market returns, our ultimate goal is to make predictions based on investors sentiments with respect to the distribution of expected returns. Moreover, the conditional quantile function of realized stock returns is of intrinsic interest in the finance literature.

Based on the methodology developed in section 2 and 3, I estimate the uniform bands for the QR coefficient of investor sentiments. The baseline model is the conditional quantile regression model for stationary time series data focusing on the S&P 500 index market.
The major result is to compare the differences in investment sentiments correlations during recessions, average returns and expansions.

4.1 Data

The data is from several resources including American Association of Individual Investors (AAII), Michigan Survey of Consumers, the Survey of Consumer Finances and CRSP. After matching the S&P 500 index with investor sentiments data, I compare the real stock returns with investor sentiments. The advantage of data from the American Association of Individual Investors is that AAII provides a longer range high frequency measure over a pool of individual investors.

Investor expectation measures are derived from the bull-bear spread, calculating the percentage of individual investors who are bullish, bearish, and neutral on the stock market for the next six months. Individuals are polled from the ranks of the membership on a weekly basis (July 24, 1987- Feb 20, 2014). Only one vote per member is accepted in each weekly voting period. Investor expectation is represented by the bull-bear spread, the difference between the percentages of bullish investors and bearish investors.

Figure 1 shows the density of Bull-Bear Spread as a measure of investor expectation. The bull-bear spread is approximately normally distributed with mean 0.08 and standard deviation of 0.19. Comparatively, the real distribution of stock market is slightly skewed to the right with extreme left fat tail, reflecting the 2007 financial crisis. The data set covers 1987-2014 weekly data of 1632 observations. $R$ is calculated as the log difference of current S&P 500 index compared to the previous week. $ICE$ is the Index for the Consumer Expectation from Michigan Survey on Consumers. $CAY$ is the model-based expected return from Consumption, Asset Wealth and Labor Income (Jurado et al, 2013). $VIX$ is the volatility index of S&P 500 stock index from Chicago Fed Database.

It’s worthwhile to point out that investor sentiments $Spread$ reached its minimum of -0.54 on Oct 19, 1990; second least -0.51 on Mar 5, 2009; maximum of 0.63 on Jun 26, 2003; second
best of 0.62 on Jan 6, 2000. These dates indicate the corresponding economic events such as financial crises and economic booming. Consistent with Greenwood and Shleifer (2014), there is a significant negative correlation between investors expectation and model-based expected returns.

The data cleaning process requires the weekly, monthly or quarterly data match as the available data sources have different starting points and frequencies. We treat the survey of investor sentiments as a direct measure of expected returns and find the positive correlation with the realized returns in the S&P 500 stock market in six months. The strong negative correlation between investor expectations and macro uncertainty shows that investors take macro fundamentals and risks seriously in making investment expectations.

Both spread and ICE (Index of Consumer Expectation) are asymptotically normal distributions with a slightly fat tail. The observations are equally distributed over quantiles. ICE survey is carried on a monthly basis while spread is calculated weekly based on the on-line survey responses. However spread has a better property for the quantile analysis and is used afterwards for the quantile implementation.

The research questions of interest are in two parts: a) How do investors’ expectations on the stock market relate to the real S&P500 stock market performance? b) Do such correlations vary across the conditional distribution of realized returns of the stock market? Is the investor expectation consistent with realized returns of stock markets and expected returns all the time?

Denote $E_t$ as the bull-bear spread, an instrument measure of investors expectation from AAII survey. Denote $R_{t+k}$ as the $k$-time ahead realized stock market return based on the daily S&P500 index. I pick six months ahead as the baseline model as the survey question asks each individual investor his/her expectation of the stock trend in the next six months. $P_{t+k-1}$ is the S&P500 index at time $t + k - 1$ matching the weekly spread.

$$Q_{R_{t+k}}(\tau) = \alpha(\tau) + \beta(\tau)E_t, \quad R_{t+k} = \log P_{t+k} - \log P_{t+k-1}$$ (10)

From the scatter plot of investor expectation and returns with the stock return, there is
a change in slopes between bullish and bearish stock markets. The disadvantage of OLS is that such changes might be highly driven by the outliers and sub sampling eliminates the full information about the market. The correlation with realized stock market is stronger during extreme cases which highlights the importance of quantile analysis. The plot of the real S&P500 stock market returns and investor expectations indicates that the data sample is uniformly distributed in the dimension of spread between -0.5 and 0.5.

4.2 Results

Figure 2 is the traditional quantile regression model with point-wise confidence bands. The intercept part follows the positive slope trend for the quantile estimates while the coefficient of the investor sentiments changes abnormally with a downward sloping. The OLS results are shown in the dashed straight line and the dotted lines give the upper and lower bound for the OLS estimator at the 95% confidence level. However, the quantile estimators are different from the OLS estimator for lower and upper quantiles.

The lower quantile \([0.05, 0.15]\) represents the realized stock returns are at the 5% to 15% left side of the whole distribution of returns. It can be interpreted as the stock market is bearish or even in crisis. The upper quantile \([0.8, 0.95]\) represents the realized stock returns are at the 80% to 95% left side of the whole distribution of returns. It can be interpreted as the stock market is bullish and in expansions. To compare the difference between the two scenario requires the confidence bands to be uniform over quantiles.

Figure 3 is the uniform confidence bands of quantile regression slope coefficients of investor sentiments. The black line is the quantile regression estimator. For different quantiles of the conditional distribution of stock returns, the correlation varies and the confidence bands changes. The inner yellow band is the uniform confidence interval of the estimator at the 90% nominal level. The outer red band is at the 95% nominal level.

The confidence bands are typically wider near the tails of the conditional distribution. The changes in signs of the correlation between investor sentiments and realized returns are
still significant. It’s unexpected that in the upper tails of the conditional distribution, there
is a negative relationship significantly. In the booming times of the S&P 500 stock market,
the more optimistic the investor sentiments are, the lower the future returns would be.

In Table 2, for each given nominal level \( \alpha \), the upper interval is the traditional point-wise
confidence band and the lower one is the uniform confidence band. For given quantile \( \tau \),
there is a unique QR estimator. I find three properties the results support: (1) Uniform
confidence bands are always larger than the point-wise confidence bands; (2) Uniform bands
are larger at the tails compared with the median; (3) The difference between uniform and
point-wise confidence bands gets smaller as \( \tau \) goes to 0.5.

5 Robustness checks

To justify the use of the parametric linear quantile regression model on investor sentiments,
this paper conducts two robustness checks on both the model assumptions and the results.
It tests whether the variables of interest are stationary. It also rules out the possible inter-
pretation that the difference in quantile estimators comes from the structural changes.

5.1 Unit Root Test of Investor Sentiments

First, we conduct the unit root test on the investor sentiments. The last column of Table 3
shows the statistic results of the test. Both the investor sentiments and weekly stock returns
are stationary based on the unit root test. Therefore, the first assumption in the section 2
is satisfied.

Second, we test the serial correlation in stock returns. The Durbin-Watson statistics and
Cochrane-Orcutt AR(1) regression two-step estimators are shown in Table 3. The D-W tests
indicates that the null hypothesis of no serial correlation could not be rejected. White test on
heteroskedasticity of the error term shows that quantile regression method has the advantage
over OLS on this specific empirical application, focusing on the analysis of heterogeneity in
the financial markets.

5.2 Structural Change Test for Quantile Regression

Based on Qu(2008), a cumum test for structure breaks is carried out to distinguish the comparison of differences between quantiles from the structure change. It evaluates CUSUM tests based on the gradient vectors and regression coefficients for nearly stationary processes with martingale difference dependence structure. The test statistics is

\[
SQ_{\tau} = \sup ||(\tau(1 - \tau))^{-1/2}(H(\hat{\beta}(\tau)) - H_0(\hat{\beta}(\tau)))||_{\infty},
\]

where \(||.||_{\infty} = \max(|z_1|, ..., |z_k|)\) for a vector \(z = (z_1, ..., z_k)\).

In Table 4, I find that there is no structural change for each quantile from 0.2 to 0.8. The difference between the two quantiles doesn’t indicate a structure change so that the comparison holds for the uniform bands.

6 Conclusion

I have developed the uniform confidence bands for the linear quantile regression estimator in the time series setting. The inference procedure allows for serially correlated errors. By applying this method, I analyze the weekly financial survey data of the investor sentiments in a quantile regression model. Information of investment behaviors is reflected from investor sentiments. The responses from investor sentiments show asymmetric pattern in the right and left tails of the return distribution. The quantile regression coefficient of investor sentiments on returns decreases as quantile increases.

Future research directions would be 1) to study the stock market volatility with respect to investor expectation variation; 2) to learn the dynamics of investor sentiments: the speed for market inefficiency to be resolved in weekly updated data; 3) to test rationality in quantiles: case study for expansions and recessions; 4) to solve the self-selection issue in investor survey data.
References


Tables

Table 1: Simulated coverage probabilities under different error terms

<table>
<thead>
<tr>
<th></th>
<th>5%</th>
<th>10%</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>n=250</td>
<td>n=500</td>
</tr>
<tr>
<td>Case I</td>
<td>0.958</td>
<td>0.969</td>
</tr>
<tr>
<td>Case II</td>
<td>0.949</td>
<td>0.951</td>
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Table 2: QR estimates and confidence bands comparison (multiplied by 100)

<table>
<thead>
<tr>
<th>τ</th>
<th>QR estimate</th>
<th>alpha=0.1</th>
<th>alpha=0.05</th>
<th>alpha=0.01</th>
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<tbody>
<tr>
<td>0.1</td>
<td>0.725</td>
<td>(0.044,1.41)</td>
<td>(-0.0866,1.54)</td>
<td>(-0.342,1.79)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-0.318,1.77)</td>
<td>(-0.437,1.89)</td>
<td>(-0.531,1.98)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.274</td>
<td>(-0.141,0.689)</td>
<td>(-0.221,0.769)</td>
<td>(-0.376,0.925)</td>
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<tr>
<td></td>
<td></td>
<td>(-0.377,0.925)</td>
<td>(-0.431,0.979)</td>
<td>(-0.544,1.09)</td>
</tr>
<tr>
<td>0.25</td>
<td>0.182</td>
<td>(-0.171,0.534)</td>
<td>(-0.238,0.601)</td>
<td>(-0.370,0.733)</td>
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<tr>
<td></td>
<td></td>
<td>(-0.346,0.709)</td>
<td>(-0.427,0.79)</td>
<td>(-0.510,0.874)</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.230</td>
<td>(-0.409,-0.051)</td>
<td>(-0.443,-0.017)</td>
<td>(-0.510,0.051)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-0.587,0.128)</td>
<td>(-0.621,0.161)</td>
<td>(-0.675,0.216)</td>
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<tr>
<td>0.75</td>
<td>-0.399</td>
<td>(-0.672,-0.127)</td>
<td>(-0.724,-0.075)</td>
<td>(-0.827,0.028)</td>
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<tr>
<td></td>
<td></td>
<td>(-0.818,0.019)</td>
<td>(-0.862,0.063)</td>
<td>(-0.935,0.136)</td>
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<tr>
<td>0.8</td>
<td>-0.568</td>
<td>(-0.909,-0.227)</td>
<td>(-0.975,-0.162)</td>
<td>(-1.10,-0.034)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-1.17,0.031)</td>
<td>(-1.23,0.094)</td>
<td>(-1.31,0.176)</td>
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<tr>
<td>0.9</td>
<td>-0.884</td>
<td>(-1.43,-0.337)</td>
<td>(-1.54,-0.232)</td>
<td>(-1.74,-0.027)</td>
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<td></td>
<td></td>
<td>(-1.73,-0.041)</td>
<td>(-1.79,0.026)</td>
<td>(-1.95,0.184)</td>
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Table 3: Robustness checks

<table>
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<tr>
<th>Variable</th>
<th>Dependent spread</th>
<th>Durbin-Watson Statistic</th>
<th>Cochrane-Orcutt</th>
<th>White</th>
<th>Unit-Root</th>
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<tr>
<td>Independent</td>
<td>$R_d$</td>
<td>2.091</td>
<td>1.996</td>
<td>1.40</td>
<td>-36.435</td>
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<tr>
<td></td>
<td>$R_k$</td>
<td>2.231</td>
<td>1.983</td>
<td>33.60</td>
<td>-42.665</td>
</tr>
<tr>
<td></td>
<td>$R_{ak}$</td>
<td>2.231</td>
<td>1.983</td>
<td>34.78</td>
<td>-42.857</td>
</tr>
<tr>
<td></td>
<td>$R_m$</td>
<td>0.066</td>
<td>2.082</td>
<td>31.11</td>
<td>-4.061</td>
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Table 4: Structural change tests in quantiles based on a single conditional quantile function

<table>
<thead>
<tr>
<th>Quantile</th>
<th>0.2</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.8</th>
</tr>
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<tr>
<td>SQ spread</td>
<td>1.344397</td>
<td>1.048882</td>
<td>1.224562</td>
<td>1.040471</td>
<td>1.440827</td>
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<tr>
<td>SQ $R_m$</td>
<td>1.225726</td>
<td>1.08149</td>
<td>1.073486</td>
<td>0.7783645</td>
<td>0.5978931</td>
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<tr>
<td>SQ $R_d$</td>
<td>1.187539</td>
<td>1.044182</td>
<td>1.231745</td>
<td>0.7904821</td>
<td>0.9935286</td>
</tr>
<tr>
<td>Critical values</td>
<td>1.520701</td>
<td>1.520701</td>
<td>1.520701</td>
<td>1.520701</td>
<td>1.520701</td>
</tr>
<tr>
<td># of changes</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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Figure 1: Histogram of investor sentiments: Bull-Bear Spread
Figure 2: Quantile regression of investor sentiments $BBspread$ on S&P500 stock returns
Figure 3: Uniform confidence bands for the QR slope coefficient of investor sentiments
Appendix A. Technical Details

Proposition 1 Proof. Under assumptions 1-5, we proceed in line of Lemma 2 of Qu (2008). Since \( ||S_T(\tau, \beta(\tau))|| \leq ||T^{-1/2} \sum_{t=1}^{T} x_{t} 1(y_t - x_{t} \beta(\tau))|| \leq pT^{-1/2} \max ||x_t|| \rightarrow^p o_p(1) \) (A1). Therefore, our goal is reduced to prove that \( \sqrt{T} (\hat{\beta}(\tau) - \beta_0(\tau)) = O_p(1) \) holds, which is equivalent to show that \( \forall \epsilon > 0, \exists K_0 > 0, T_0 > 0, \eta > 0, \text{s.t.} \) if \( ||\sqrt{T}(\hat{\beta}^*(\tau) - \beta_0(\tau))|| > K_0, \Rightarrow P(\inf_{\tau} ||S_T(\tau, \beta^*(\tau))|| < \eta) < \epsilon, \forall t > T_0 \), where \( \beta^*(\tau) = \beta_0(\tau) + T^{-1/2}b \times \epsilon, b = ||\beta^*(\tau) - \beta_0(\tau)||, e = (\beta^*(\tau) - \beta_0(\tau))/||\beta^*(\tau) - \beta_0(\tau)|| \in R^d \) which is equivalent to show that

\[
P(\inf_{\tau} ||S_T(\tau, \beta^*(\tau))|| < \eta) \leq P(\inf_{\tau} ||e'S_T(\tau, \beta_0(\tau) + T^{-1/2}be)|| < \eta)
\]  

(12)

For a non-decreasing function of \( b: S_T(\tau, \beta_0(\tau) + T^{-1/2}be) \), we can split the above probability into two parts based on the conditional probability theory and sup/inf inequality. By Cauchy-Schwarz inequality, we have the following inequalities:

\[
P(\inf_{\tau} ||e'S_T(\tau, \beta_0(\tau) + T^{-1/2}be)|| < \eta) \leq P(\inf_{\tau} ||e'S_T(\tau, \beta_0(\tau) + T^{-1/2}be)|| < \eta, \inf_{\tau} ||e'S_T(\tau, \beta_0(\tau)) + e'H_0K_0e|| \geq 2\eta) + P(\inf_{\tau} ||e'S_T(\tau, \beta_0(\tau)) + e'H_0K_0e|| < 2\eta) \leq Q_1 + Q_2
\]

where

\[
Q_1 = P(\sup_{\tau} ||e'S_T(\tau, \beta_0(\tau)) + e'H_0K_0e - e'S_T(\tau, \beta_0(\tau) + T^{-1/2}be)|| \geq \eta)
\]

(13)

\[
Q_2 = P(\sup_{\tau} ||e'S_T(\tau, \beta_0(\tau))|| > K_0 \inf ||e'H_0e|| + 2\eta)
\]

(14)

To show the above two probabilities can be arbitrarily small for large \( T \), we employ two properties of the subgradient term \( S_T(\tau, \beta) \) based on the previous assumptions.

First of all, under A1-5, the process \( S_T(\tau, \beta) \) is stochastically equi-continuous over \( \tau \). Because \( S_T(\tau, \beta_0(\tau)) = T^{-1/2} \sum_{t=1}^{T} x_{t} [1(F_t(y_t \leq \tau) - \tau) \] with \( F_t(.) \) absolute continuous and strictly increasing almost everywhere and \( x_t [1(F_t(y_t \leq \tau) - \tau) \] is a sequence of vector martingale differences.

Secondly, \( S_T(\tau, \beta_0(\tau)) \) satisfies for an arbitrary compact set \( D \) in \( R^d \),

\[
\sup_{\tau \in [\omega, 1-\omega]} \sup_{\epsilon \in D} ||S_T(\tau, \beta_0(\tau) + T^{-1/2}\epsilon) - S_T(\tau, \beta_0(\tau)) - T^{-1/2} \sum x_t(F_t(x_{t}(\beta_0(\tau) + T^{-1/2}\epsilon)) - \tau)|| = o_p(1).
\]

(15)

With the above properties through Taylor expansion, (13) can be made arbitrarily small as \( T \) goes infinity. Because \( S_T(\tau, \beta_0(\tau)) \) is relatively compact, for any \( \epsilon > 0 \), there always
exists a $K$ such that when $T$ is large enough, $P(\sup_\tau ||e'S_T(\tau, \beta_0(\tau))|| > K) < \epsilon$. Pick $K_0$ as inf $e'H_0e$ is bounded away from zero such that $K_0 \inf e'H_0e > K + 2\eta$ to complete the proof.

We can also establish

$$Z_T(\tau) = \sqrt{T}(\beta^*(\tau) - \beta_0(\tau)) = O_p(1).$$

By applying mean value theorem, we have reached the targets. The above results hold uniformly in $\tau$ and valid for sub samples as well.

\[ \blacksquare \]

Proposition 1 doesn't require a specific form for the error term in the linear quantile regression model. It has been used in the proof of theorem 2 and 3. The following remark shows that with additional assumption 6, the variance of the QR estimator can be used to derive the uniform confidence bands.

**Remark** Under Assumption 1-6 and for large $T$,

$$\hat{\sigma}^2(\tau) = \text{plim}_{T \to \infty} \sigma^2(\tau) = \tau(1 - \tau)H_0^{-1}J_0H_0^{-1},$$

(16)

$\hat{\beta}(\tau)$ is a vector that consists of quantile regression estimates for $\beta_0(\tau)$ uniformly in $\tau \in (\omega, 1 - \omega)$ using the full sample. $\hat{\sigma}^2(\tau)$ is a consistent estimate of the variance of $\sqrt{T}\hat{\beta}(\tau). H_0 = \lim_{T \to \infty} T^{-1} \sum_{t=1}^T f_t(F_t^{-1}(\tau))x_t'x_t'$. $\tau \in [\omega, 1 - \omega] \subset (0, 1)$ because uniform Bahadur representation holds only on a compact sub interval on $[0, 1]$. The tail behavior of the quantiles is hard to control. Assumptions made for tails require some form of extreme value theory and standard Brownian bridge theory.

Based on the previous lemma and the definition of the ’gradient’ $S_T(\tau, \beta)$, we denote

$$H_0(\hat{\beta}(\tau) - \beta_0(\tau)) \to^d N(0, \tau(1 - \tau))$$

(17)

$H_0(\tau)$ is defined under the true parameter with no mis-specification and $X = (x_1', x_2', ..., x_T')'$. One elegant property inside $S_T$ is that $\phi_\tau(y_t - x_t'\beta(\tau))$ is a sequence of independent binary random variables with mean zero and variance $\tau(1 - \tau)$ under the assumption of independent normal error term distribution.

Therefore under A1-5 and some regularity conditions, $H_T(\beta_0(\tau))$ converges to a limiting
distribution that is nuisance parameter free. For the quantile regression estimator \( \hat{\beta}(\tau) \) obtained using the full sample, \( H_T(\hat{\beta}(\tau)) = (X'X)^{-1/2} \sum_{t=1}^T x_t \phi_{\tau}(y_t - x'_t \hat{\beta}_0(\tau)) \). The preceding quantity converges to a summation of degenerate distributions under correct specification and monotonic order because the uniformly consistent quantile estimator \( \hat{\beta}(\tau) \) serves as a good estimate of its population value.

Therefore, \( H_T(\beta(\tau)) \) has the same stochastic order as its population counterpart. For the estimated residuals in the quantile regression, the statistic is away from the true quantile under mis-specification. Based on A4 and the property of degenerate distributions, \( \text{plim}_{T \to \infty} \hat{S}_T(\tau) = \tau(1 - \tau)H_0^{-1}J_0J_0^{-1}. \)

To show the weak convergence of the finite dimensional distributions and tightness, we have that under assumption 1-6, following proposition 1 and lemma 1, we already show that \( \sqrt{T}(\hat{\beta}(\tau) - \beta_0(\tau))H_0 = \hat{S}_T(\tau, \beta(\tau)) - S_T(\tau, \beta_0(\tau)) + o_p(1) \) and \( S_T(\tau) \Rightarrow J_0^{1/2}W(\tau) \) uniformly in \( \tau. \)

Under the martingale difference assumption of \( ((\tau - 1(F_t(y_t|X_t) < \tau))x_t,X_t), 1 \leq t \leq T, \) we have
\[
E(S_T(\tau_1)S_T(\tau'_2)) = (\tau_1 \land \tau_2 - \tau_1 \tau_2)T^{-1} \sum_{t=1}^T E(x_t x'_t) \to_p (\tau_1 \land \tau_2 - \tau_1 \tau_2)J_0 \quad (18)
\]

As \( S_T(\tau) \) is a sequential weighted empirical process with no heteroskedasticity, the tightness follows from Bai (1996). The finite dimensional convergence to a normal distribution follows from the central limit theorem for a martingale difference sequence and the Cramer-Wold device. Therefore our calculation of the covariance matrix holds for the limiting Gaussian process. This completes the proof.

Corollary in section 3 follows by equicontinuity of the uniform quantile estimator and Bahadur representation. As the process of \( S(\tau) \) is estimated consistently uniformly in \( \tau \in (\omega, 1 - \omega) \), the result follows by the argument to the proof of Corollary 2 in Qu and Yoon (2015).

Note that \( W_\alpha \) is the \( (1 - \alpha) \) percentile of \( \sup_\tau |W(\tau)| \). By continuous mapping theorem, we have \( P(Q(\tau|x) \notin C_\alpha \text{ for some } \tau \in (\omega, 1 - \omega)) = P(|Q(\tau|x) - \hat{\beta}(\tau)|). \)
Lemma 1  Under Assumption 1-5, S1 and for large T,

\[ \sqrt{T}(\hat{\beta}(\tau) - \beta_0(\tau)) - M_0(T, \tau)^{-1} S_T(\tau, \hat{\beta}(\tau)) = O_p(T^{-1/4} \log^{3/2} T). \]  \hspace{1cm} (19)

uniformly in \( \tau \in (\omega, 1 - \omega) \). \( \hat{\beta}(\tau) \) is the estimate of \( \beta_0(\tau) \) using the full sample. \( S_T \) is defined as the full sample gradient term of the QR estimating process.

\[ M_0(T, \tau) = T^{-1} \sum_{t=1}^{T} f_t(i, \tau|G_0)x_i'x_i'. \]  \hspace{1cm} (20)

Proof: Suppose conditions of Lemma 3.1 holds. Let \( \delta_i \) be a positive number array such that \( \delta_n \to 0 \). Then based on Wu and Zhou (2016) lemma 1.1, we have the corresponding property of \( M(T, \tau) \) for a given \( \tau \).

Following the proof of lemma 2.1 while replacing \( H_0 \) by \( M_0 \), we take into account the serial correlation. Here we can derive the property of \( M_0 \) following Wu(2007). Let \( \{\delta_i\}_{i=1}^n \) be a number array such that \( \delta_n \to 0 \). Then there exists a set \( w_n \) such that, for \( 0 < t \leq n^3 \), \( M \) be large enough constant, \( \lim_{n \to \infty} P(W_n) = 0 \), we have \( E\{exp(t \sup_{|\theta| \leq \delta_n} |M_n(\theta) - M_n(0)|f(W_n))\} \leq M exp(4t \sqrt{K_1 n \delta_n} \log n) \).

As \( \delta_n \to 0 \),

\[ \| \sup_{|g| \leq \delta_n} |N_n(g) - N_n(0)| |s \leq C \delta_n(\sqrt{ns \log^{2p+2} n} / (1 - \chi^{1/2}) + n^{p+1} n^{1-t_s \log n^{2s}}). \]

Following Taylor expansion, gradient properties and error term assumptions, we could show the difference is \( O_p(T^{-1/4} \log^{3/2} T) \). Under the conditions of Lemma 3.1, for all \( i \in [0, r_1] \), \( t, s \in (c_i, c_{i+1}] \), and a constant \( \eta = \frac{p+1}{v(p+1)+1} \) with \( v = \frac{4p+4}{2p+1} \), we have \( \|G_0(t, F_0) - G_0(s, F_0)\| \leq C |t - s|^\eta \) and \( \|G_0(t, F_0)\|_4 \leq C |t - s|^\eta \). When \( p \) increases, the assumption S1 implies better smoothness and moment conditions for the error term \( e_i \) and \( \eta \) will get closer to \( 1/2 \).

Assign \( C \) to be any sufficiently large number for the convenience of the proof. Without loss of generality, by Cauchy Inequality, \( ||\phi(e_i(t)) x_i(t)\|_4 \leq C ||x_i(t)\|_{4(p+1)} < \infty \). Recall that \( ||x||_p = (\sum |x|^p)^{1/p} \) as the definition of \( l_p \) norm of \( x \). Therefore we show that up to \( l_4 \)-norm, the gradient vector is bounded above from infinity. For any \( t, s \in (b_{w(i)}, b_{w(i)+1}] \),

25
\[ ||\phi(e_i(t))x_i(t) - \phi(e_i(s))x_i(s)|| \leq ||\phi(e_i(t))x_i(t) - \phi(e_i(t))x_i(s)|| + ||\phi(e_i(t))x_i(s) - \phi(e_i(s))x_i(s)||.\]

Each term is bounded based on Cauchy Inequality and Markov Inequality. \[ ||\phi(e_i(t))x_i(t) - \phi(e_i(s))x_i(s)|| \leq C||x_i(t) - x_i(s)||_{4(p+1)} \leq C|t-s| \] for constant C large enough. \[ ||\phi(e_i(t))x_i(s) - \phi(e_i(s))x_i(s)|| \leq C||\phi(e_i(t)) - \phi(e_i(s))||_{2(2p+2)/p+1)} |t-s|.\]

**Proposition 2** Under assumption S1, on a richer probability space, there exists a p-dimensional zero-mean Gaussian process \( U_{\tau}(t) \), with covariance function \( \gamma(t, s) = \int_{0}^{\min(t,s)} \Sigma_{\tau}^2(r) dr \), such that
\[
|S_T(\tau, e_t) - U_{\tau}| = o_p(T^{-1/4} \log^2 T). \]
where \( S_T(\tau, e_t) = T^{-1/2} \sum_{t=1}^{T} x_t \phi(\tau) \) and \( \Sigma_{\tau}^2(t) = \sum_{k} \text{Cov}(G_{\tau}(t, F_{0}), G_{\tau}(t, F_{k})) \), \( t \in (0,1) \). Here we assume \( \Sigma_{\tau}^2(t) \) has the smallest eigenvalue bounded away from 0 on \([0,1]\).

**Proof.** Let C be a sufficiently large number which may vary from line to line. Recall that by Cauchy Inequality, \( ||\phi(e_i(\tau))x_i||_{4} \leq C||x_i||_{4(d+1)} < \infty \). To show \( S_T(\tau, e_t) \Rightarrow U_{\tau} \) on compact set over \((0,1)\), we apply Proposition 5 from Zhou(2013), which states that under assumptions S1-2, on a possibly richer probability space, there exist iid standard normal random variables \( V_1, ..., V_n \) such that
\[
\max_{1 \leq i \leq n} \left| \sum_{j=1}^{i} e_j - \sum_{j=1}^{i} \sigma(t_j V_j) \right| = o_p(n^{1/4} \log^2 n) \quad (21)
\]

Note that the time series \( e_i \) is composed of stationary time series in the sense of Zhou and Wu (2009). The above proposition is a straightforward extension of that of Corollary 1 of Wu and Zhou(2011). Therefore, we can apply Proposition 1.2 in Wu and Zhou (2014) to prove equation (27). When \( d \) increases, the assumption S-1 imply better smoothness and moment conditions for \( e_t \), which forces \( \eta = (d+1)/[v(d+1) + 1] \) getting closer to 1/2. 

**Theorem 1** Proof: Under Assumption A1-6 and S1, we would have the following property hold

\[ |\hat{\beta}(\tau) - \beta_0(\tau)| \leq_p T^{-1/2} \log T \quad (22)\]
As $\hat{\beta}_T(\tau)$ is weakly consistent, we could derive theorem directly from Lemma 1 combining the following property. Define $\Theta = \{ \theta : |\theta| \leq Cn^{-1/2} \log^4 n \}$. Then,

$$
\sup_{1 \leq j \leq n} \sup_{\theta' \in \Theta} |K_j(\theta) - K_j(\theta')| = O_p(n^{1/4} \log^3 n).
$$

(23)

Here $K_j(\theta)$ is defined by $K_j(\theta) = \Omega_j(\theta) - E(\Omega_j(\theta))$ and $\Omega_j(\theta) = \sum_{i=1}^j \phi(e_i - x'_i \theta) x_i$. \{M_j, j = 1, ..., n\} is a martingale with respect to the filtration $\varphi_i$. Write for $j = 1, ..., n, M_j(\theta) = \sum_{i=1}^j (\phi(e_i - x'_i \theta) x_i - E[\phi(e_i - x'_i \theta) x_i | \varphi_i])$. $N_j(\theta) = \sum_{i=1}^j \{E[\phi(e_i - x'_i \theta) x_i | \varphi_i] - E[\phi(e_i - x'_i \theta) x_i]\}$. Then $K_j(\theta) = M_j(\theta) + N_j(\theta)$.

Based on lemma 1.1 in Wu and Zhou (2014), let $\delta_i$ be a positive number array such that $\delta_n \to 0$. Then $\max_{1 \leq j \leq n} \sup_{|\theta| \leq \delta_n} |M_j(\theta) - M_j(0)| = O_p(\sqrt{\tau_n(\delta_n)} \log n + n^{-3})$.

$$
\max_{1 \leq j \leq n} \sup_{|g| \leq \delta_n} |N_j(g) - N_j(0)| = O_p(\sqrt{n} \delta_n).
$$

Therefore we can derive the boundedness of $K_j(\theta)$ and the above property of $K_j(\theta)$.

Assume under A1-6 and S1, it can be shown that

$$
\max_{0 \leq \tau \leq 1} \left| \frac{1}{n} \sum_{i=1}^{\lfloor sn \rfloor} (f_w(i/n, 0|G_0)x_i x'_i - E(f_w(i/n, 0|G_0)x_i x'_i)) \right| = O_p(n^{-1/2} \log^{7/2} n)
$$

Recall the definition of $\Lambda_c_n$ in section 3. By the martingale decomposition technique we applied in Lemma 1, we have that $\max |H_0(\tau) - \Lambda_c_n(\tau, t)| = o_p(T^{-1/2})$.

Based on the proposition of $U(\tau)$, we have the limiting distribution of QR estimator uniformly over $\tau$. The gradient-based process $\Psi_m(t)$ is a uniformly consistent estimator of $U_\tau(t)$, where $\Psi_m = \sum_{i=1}^n (m(n-m+1))^{-1/2}(\hat{\omega}_{i,m} - \hat{\omega}_{i,1}) R_i$. $\omega_{j,m} = \sum_{r=j}^{n-m+1} \phi_r(\hat{e}_r(\tau)) x_r$ and \{R_i\}_{i=1}^n are i.i.d. standard normals which are independent of $\{\varphi_i\}_{i=\infty}$. $\hat{\Lambda}_c_n(t, \tau)$ converges to $H_0(\tau)$ in probability and $\hat{\Psi}_{m,n}(t)$ are consistent estimate of $U(\tau)$ with the uniform topology.

The uniform confidence bands under serially correlated errors changes in the derivation of the critical value. Select m and c_n based on minimum volatility (MV) method in section 4 of Zhou (2003), first advocated in Politis et.al (1999); Step 2: Generate B times (e.g. B=10000) $\{\Psi_m\}$ for sample size T, and get $\hat{\Lambda}_{c_n}(i, \tau), i = 1, ..., T$; Step 3: Calculate $E_b(\tau) = \hat{\Lambda}_{c_n}^{-1}(T, \tau) \Psi_m(\tau)$ for every given $\tau$; Step 4: Let $E_b = \sup_{\tau} E_b(\tau)$. Let $E_{(1)} \leq E_{(2)} \leq ... \leq E_{(B)}$ be the order statistics of $E_b$. Then $E_{(1-\alpha)B}$ is the level $\alpha$ critical value for the QR estimator coefficient uniform confidence bands.
The simulation process follows the theorem and through martingale decomposition technique, we have Lemma 1. Take $\theta = \hat{\beta}(\tau) - \beta_0(\tau)$. The proposition of $U(\tau)$ on $(0, 1)$ with the uniform topology follows a similar argument in Wu and Zhou (2014) Proposition 1.4. We can also check that when the dependence is geometrically decaying.

**Appendix B. Survey Details**

Question: Who are these potential investors taking the survey (public/inner information accessible)?

Answer: ICE, the monthly Survey of Consumers is an ongoing nationally representative survey based on approximately 500 telephone interviews with adult men and women living in households in the coterminous United States (48 States plus the District of Columbia). The sample is designed to maximize the study of change by incorporating a rotating panel sample design in an ongoing monthly survey program. For each monthly sample, an independent cross-section sample of households is drawn.

Spread: individual investors armed with effective investment education materials and a bit of dedication could outperform the popular market averages. Over thirty years later, the 150,000 members of American Association of Individual Investors report investment returns that are consistently higher than those of the stock market as a whole. Further AAII’s Shadow Stock Portfolio (a real-money portfolio used to teach members about investing) has lower risk scores and better returns than the S&P 500 for the last 10 years.

Lists of the survey questions:

1) Looking ahead, do you think that a year from now you (and your family living there) will be better off financially, or worse off, or just about the same as now?

2) For business conditions in the country as a whole, do you think that during the next twelve months we’ll have good times financially, or bad times, or what?

3) Looking ahead, which would you say is more likely, that in the country as a whole we
will have continuous good times during the next five years or so, or that we will have periods of widespread unemployment or depression, or what?

Return R is the measure of realized return of S&P500 index. Denote \( R^d_t = \ln P_t - \ln P_{t-1} \) as the current daily return using the today’s closed S&P 500 in log minus yesterday’s closed index. Denote \( R^k_t = \ln P_t - \ln P_{t-k} \) where the current weekly return is matched with the weekly survey of investor sentiments release day using the difference of today’s closed S&P 500 in log and the corresponding previous week’s closed index. Some survey release days are in the weekends (holiday or special occasions), therefore the nearest opening day index is employed.