Abstract

The stylized facts in the historical dynamics of U.S. Treasury bond yields – a trend in long-term yields, business cycle movements in short-term yields, and a level shift in yield spreads – reflect key features that the pricing kernel of any equilibrium model should have. This paper presents an equilibrium asset pricing model with subjective expectations to jointly explain these puzzling facts. The trend is generated by subjective expectations of long-run GDP growth and inflation, which share similar patterns to the neutral rate of interest ($r^*_t$) and trend inflation ($\pi^*_t$) estimates in the literature. Cyclical movements in yields and spreads are mainly driven by expected short-run GDP growth and inflation. The less-frequent inverted yield curves (and steeper curve) observed after the 1990s are due to the recent secular stagnation and procyclical inflation expectations.

Keywords: Subjective expectations, underreaction, bond yields, neutral rate of interest, trend, cycle, inverted yield curves, secular stagnation

JEL Classification: G00, G12, E43

1. Introduction

Some basic properties of the stochastic discount factor (SDF) have been established in the literature. Examples include the volatility bound of the SDF (Hansen and J-
gannathan, 1991) and the permanent-transitory decomposition of SDF (Alvarez and Jer-
mann, 2005; Hansen and Scheinkman, 2009; Hansen, 2012). Given that the SDF from
any equilibrium model has direct implications for yield curves, the dynamics of the Treas-
ury yield curve should tell us, in addition to these basic properties, what a good SDF
should look like from a historical perspective. Figure 1 below shows some salient fea-
tures in the data: (1) a hump-shaped trend in the yields, (2) business cycle (cyclical)
movements in short-term yields and in the spreads between long- and short-term yields,
(3) more-frequent and deeper inverted curves (accompanied by more-frequent recessions)
pre-1990s than post-1990s, and (4) a positive yield spread on average, but shifting higher
after the 1990s.

Most equilibrium term structure models are designed to interpret and quantify the
means, volatilities, and average positive spreads in yields. Given the standard stationarity
assumption (which implies stationary short rates), it is hard for these models to generate
the hump-shaped trend and match the unconditional volatility in yields using model
implied short rates. As a result, both the low-frequency variations in the long-term yields
and the positive yield spread are determined to be the risk premia (see, for example,
the inflation risk premium in Piazzesi and Schneider 2007). The key assumption for
generating a positive inflation risk premium and, hence, an upward-sloping yield curve,
is that inflation is bad news for future growth. However recent studies have shown that,
following the late 1990s, inflation has switched to a good-news event for future growth.1
This fact implies a negative inflation risk premium and a downward-sloping nominal curve
in equilibrium. Figure 1 (right panel) shows the opposite: the yield spreads shifted to
an even higher level after the 1990s. Furthermore, it is hard to explain business cycle
movements in yield spreads using a risk-premium approach – moving from positive out of
recessions to negative in a late expansion stage. Finally, the third fact – the more-frequent
and deeper inverted curves for pre-1990s than for post-1990s – has been overlooked in the
literature. Given the recent concern over the association between recessions and inverted

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1See, for example, Burkhardt and Hasseltoft (2012); David and Veronesi (2013); Campbell et al.
(2017); Zhao (2020)
yield curves, we need an equilibrium interpretation for this fact. In this paper, we show that the historical dynamics of the yield curve can largely be explained by movements in the subjective expectations of GDP growth and inflation, and provide a joint-equilibrium understanding of these salient features in the data.

Without trend and cycle decomposition in the data generating process of economic fundamentals, it is difficult for conventional equilibrium models to match both low-frequency and business cycle movements in bond yields. In this model, both the GDP growth and the inflation rates are decomposed into two components: one stable and one transitory/volatile. The stable component varies less with the business cycle and is assumed to contain a random-walk state variable (capturing trend growth and inflation); while the transitory component varies greatly during economic contractions and expansions and is assumed to contain a stationarity state variable (measuring short-run deviations from the trend). The agent then forms subjective expectations on both long-run trends and short-run deviations of GDP growth and inflation.

To model subjective beliefs, we depart from rational expectation by allowing the agent to put more or less weights on data observations than a rational agent would when he/she updates posterior beliefs, which can lead to over- or under-react to macro news. Similar to diagnostic expectations (Bordalo et al., 2020a), the case of over-reaction is based on the psychology foundation that the agent perceives small samples to represent their population equally well as large samples (Tversky and Kahneman, 1971). The subjective
learning gains are then estimated by matching model-implied subjective expectations with survey consensus. We find that, consistent with Coibion and Gorodnichenko (2015), the agent under-reacts to news when forms his/her subjective expectations on GDP growth and inflation. The representative agent is assumed to have a constant relative risk aversion (CRRA) utility, which implies a constant subjective risk premium. Bond price/yield movements are driven by agent’s long- and short-run subjective expectations.

First, it has long been recognized that nominal interest rates contain a slow-moving trend component (Nelson and Plosser, 1982; Rose, 1988). Recent empirical studies propose macro trends as the driving force behind this low-frequency variation. For example, Kozicki and Tinsley (2001) and Cieslak and Povala (2015a) document the empirical importance of trend inflation ($\pi_t^\ast$) for explaining the secular decline in Treasury yields since the early 1980s. Bauer and Rudebusch (2020) show that it is crucial to also include the neutral rate of interest ($r_t^\ast$, modeled as random-walk process), which has driven the downward trend in long term yields over the last 20 years. After accounting for the trend component, the term premium in these empirical models is relatively small and stationary. In most equilibrium models, however, short rates are assumed to be stationary and the majority of variations in yields is explained by certain types of risk premia.

In this model, the subjective expectations for the long-run inflation and long-run growth rates, as state variables, capture the trends in inflation and growth, and they affect the yields of all maturities equally. The model-implied $r_t^\ast$, which is a linear function of the long-run expectations, moves closely with the $r_t^\ast$ estimates in the literature. Not only does the model capture the low-frequency variations, but the model-implied $r_t^\ast$ also exhibits a moderate business cycle component. Furthermore, the posterior of the trend inflation matches the survey-based trend inflation. As a result, the hump-shaped trend in the 10-year Treasury yield (from the 1970s to the late 1990s) reflects an increase in

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2The finding of under-reaction (for growth and inflation) to news is potentially related to the Fed’s dual mandates. The agent chooses (optimally) to put less weights on inflation and growth shocks for belief updatings if she/he expects that the Fed will stabilize future growth and inflation. For other variables that are not directly related to the Fed’s mandates, for example, dividend and earning growth, agents tend to over-react to news (Bordalo et al., 2020a; Zeng and Zhao, 2021).
long-run inflation expectations before the mid-1980s and a secular decline afterwards. More recently, as inflation expectations became stabilized, the decline in the posterior for trend output growth and, hence, the $r_t^*$, has been the main driver of the downtrend in nominal yields.\(^3\) Especially, a pronounced decline in $r_t^*$ and long-term Treasury yields happened during the financial crisis periods around 2008.

Second, the subjective expectations for the short-run inflation and short-run GDP growth, as state variables, capture the cyclical movements in inflation and growth. The short-run expectations are mean reverting AR(1) processes. And they have larger impacts on the short-term yields than on the long-term yields, which implies a positive (negative) spread when the short-run beliefs are negative (positive). Therefore, the model can generate cyclical movements in the short-term yields and, hence, in the spreads.

The model-implied 1-year nominal yield and the spread between 10- and 1-year nominal yields closely track historical movements of their counterparts in Treasury bonds. Out of recessions, the short-term nominal yield starts to rise when agents begins to revise their beliefs for short-run growth and inflation upwards towards their long-run means (short-run deviations are still negative and the spread is positive), and the spread starts to shrink as these short-run expectations move towards becoming positive. This pattern continues until the late expansion stage, when the short-run growth and inflation expectations are above their long-run trends (the short-run deviations are positive now),

\(^3\)In this model, output growth is given exogenously and $r_t^*$ is driven by learning about the trend growth. Alternative interpretations include lower productivity growth, changing demographics, a decline in the price of capital goods, and strong precautionary savings flows from emerging market economies. See, for example, Summers (2014); Kiley (2015); Rachel and Smith (2015); Carvalho et al. (2016); Hamilton et al. (2016); Laubach and Williams (2016); Johansen and Mertens (2018); Christensen and Rudebusch (2019); Holston et al. (2017); Lunsford and West (2018); Del Negro et al. (2017). Caballero et al. (2008) show that the downward trend in interest rates was due to a shortage of safe assets and the increasing global imbalances. Farhi and Gourio (2019) find that rising market power, rising unmeasured intangibles, and rising risk premia, played a crucial role for the decline in real short rates over the past 30 years.
implying an inverted yield curve.\textsuperscript{4} The agent then begins to revise their short-run beliefs sharply downwards, entering a recession, and the spread switches from negative to positive.

Third, Figure 5 (right panel) shows that (1) the posteriors for short-run inflation and growth deviations moved in opposite directions before the late 1990s and in the same direction afterwards, and (2) the posteriors for both short-run inflation and growth deviations were persistently negative for most of the post-2000 period (consistent with secular stagnation; see, e.g., Summers 2014). These observations imply that short-run inflation and growth expectations drove the yields in opposite directions pre-2000, and in the same direction afterwards. Moreover, both short-run inflation and growth expectations imply positive spreads for the past two decades. Therefore, consistent with the data, the model generates more-frequent and deeper inverted curves for the period before the late 1990s than for the most-recent periods. From a monetary policy point of view, the Federal Reserve faced a trade-off between short-run inflation and growth pre-2000 and no such trade-off afterwards.\textsuperscript{5} Hence, we observe more-frequent recessions for the period before the late 1990s and less-frequent recessions (or longer business cycles) for the period afterwards.

Furthermore, the majority of model-implied nominal yield variations is due to real yield variations, not inflation variations. This is consistent with Duffee (2018) who finds that inflation news account only for 10\% to 20\% of variances of Treasury yield shocks, and we find that inflation news account for 8\% to 30\% of the model-implied nominal yield shocks. Because of CRRA utility, subjective risk premium is constant and the expectations hypothesis (EH) roughly holds under the subjective expectations. However,\textsuperscript{4}

\textsuperscript{4} The model is consistent with our conventional understanding of business cycles. From trough to peak, when the inflation and output gaps (short-run deviations in this model) are moving from negative to positive, monetary policy turns from accommodative to contractionary (short-term yields increase) according to the Taylor Rule. The difference between the standard Taylor Rule and the model-implied short rate is discussed in Section 2.3.

\textsuperscript{5} It is commonly believed that the U.S. economy was mostly hit by supply shocks before the late 1990s, and mostly hit by demand shocks afterwards, and that this generated the change in the correlation between short-run inflation and growth expectations.
the economic fundamentals are evolving according to the true data generating processes and the agent makes expectation errors. Therefore, consistent with Cochrane and Piazzesi (2005), the realized bond returns are predictable due to these expectation errors – providing an equilibrium interpretation for the empirical studies on expectational errors in Froot (1989), Piazzesi et al. (2015), and Cieslak (2018).

Finally, despite the fact that the model-implied nominal spread can stay positive or negative for an extended period of time at different phases of the business cycle, the level is almost in parallel lower than data due to the stationary assumption (mean zero) for the short-run beliefs as well as the CRRA utility. The common equilibrium explanation for the upward-sloping nominal yield curve is the inflation risk premium (Piazzesi and Schneider, 2007), where inflation is bad news for future growth and the agent prefers an early resolution to the uncertainty. Zhao (2020) shows that this approach was less effective during the past two decades when inflation switched from bad news to good news for future growth, providing an alternative worst-case belief approach through ambiguity.

We show in the appendix that this model can be extended by incorporating the intuition discussed in Zhao (2020). The ambiguity-averse representative agent (with the recursive multiple priors, or maxmin, preferences in Epstein and Schneider 2003) has in mind a benchmark or reference measure of the economy’s dynamics that represents the best estimate of the stochastic process. The reference measure in the extended model is the full stochastic environment presented in the benchmark model (including the posteriors). But the agent is concerned that the reference measure is misspecified and believes that the true measure is actually within a set of alternative measures that are statistically close to the reference distribution. Using forecast dispersion to quantify the size of the ambiguity (following Ilut and Schneider 2014), the model-implied short-rate expectations are upward-sloping under investors’ worst-case equilibrium beliefs, which generates upward-sloping nominal and real yield curves, even with a CRRA utility.
Related literature

This paper is related to a large literature on equilibrium asset/bond pricing models.\(^6\) This paper is most closely related to Piazzesi and Schneider (2007), who show the importance of the inflation risk premium in explaining the upward-sloping nominal curve in a stationary state space model (for inflation and growth). While most equilibrium bond-pricing models focus on the first/second moment and the average spread in the yields, we show that some key features in the historical dynamics of U.S. Treasury bond yields pose serious challenges to existing models, and we provide a joint equilibrium understanding of the trends, cycles, and spreads in the data.

This paper differs from previous studies along some important dimensions. First, this is the first paper that decomposes both GDP growth and inflation into two components and shows that subjective beliefs about the long-run trend drive the low-frequency variations in the yields (and in \(r_t^*\)) and that the subjective expectations of the short-run deviations from trends drive the business cycle movements in the short-term yield and, hence, in the spreads. Moreover, the subjective expectation formation process is consistent with the empirical finding of under-reaction to news for GDP and inflation. Third, this paper provides an interpretation for an important but often overlooked fact – the less-frequent inverted yield curves (and the less-frequent recessions) after the 1990s. Finally, instead of the inflation risk premium (Piazzesi and Schneider, 2007) and the real risk premium (Wachter, 2006; Albuquerque et al., 2016; Berrada et al., 2018), the upward-sloping nominal and real curves in the U.S., at least for the post-2000 periods, were partially due to persistently negative short-run inflation and growth expectations and procyclical inflation expectations for most of these periods. However, short-run inflation and growth expectations moved in opposite directions pre-2000, which makes it hard for the benchmark model to generate an average upward-sloping nominal curve (the short-run expectations and, hence, the yield spreads are stationary and mean zero). We

therefore rely on the worst-case belief approach (Zhao, 2020) to generate upward-sloping nominal and real curves that are consistent with the data.

The paper is also related to a large empirical literature that links macro information and macro trends with yield curve modeling. This paper bridges an important gap between the empirical and equilibrium yield curve literature by interpreting macro trends as posteriors for learning about long-run trend growth and inflation rates. Furthermore, the paper also provides an equilibrium interpretation for the cyclical movements in short-term yields and, hence, in yield spreads.

This paper is related to a number of papers that study the implications of ambiguity and robustness for finance and macroeconomics. Finally, this paper is also related to some recent developments wherein the implications of subjective learning in finance were investigated. For example, De la O and Myers (2020) and Bordalo et al. (2020b) show the empirically importance of subjective expectation in aggregate equity markets. Based on this model, Zeng and Zhao (2021) build a subjective expectations equilibrium model to study the term structure of equity yields and the stock-bond comovements. Nagel and Xu (2019) show how constant-gain learning about constant parameters based on Malmendier and Nagel (2011) and Malmendier and Nagel (2016) can help separate subjective and objective equity premia and explain the predictability of excess returns.

The paper continues as follows. Section 2 outlines the expectation formation process and solves the model in closed form. Section 3 describes data and shows how parameters are calibrated and estimated. Section 4 presents our empirical findings. Section 5 provides further robustness checks for the sensitivity of the results, and Section 6 concludes.

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2. Model with Subjective Expectations

In this section, we consider an endowment economy (GDP as endowment) with a representative agent who has a CRRA utility function. Both GDP growth and inflation rates are decomposed into two components: one stable component and one transitory/volatile component. The agent forms subjective expectations of long-run output growth and inflation rates from the stable components, and forms subjective expectations of short-run output growth and inflation rates from the transitory/volatile components. Equilibrium prices adjust such that the agent is happy to consume the output as an endowment.

2.1. Expectation formation

We first introduce an illustrative framework to show how the agents form their subjective expectations and then apply this idea to GDP growth and inflation in next subsection. A simple state space model highlights the intuition.

\[ y_t = x_t + \sigma \epsilon_t \]  
\[ x_{t+1} = \rho x_t + \sigma u_{t+1}, \]  

where \( y_t \) denotes the observable economic outcomes, \( x_t \) captures the latent state, and \( \epsilon_t \) and \( u_t \) are standard normal \( i.i.d. \) shocks. The rational belief updating is defined as the Bayesian updating:

\[ p(x_t|I_t) \propto p(y_t|x_t) \times p(x_t|I_{t-1}), \]  

and standard Kalman filter implies the following dynamics for the posterior belief of \( x_t \):

\[ E_t x_t = \rho E_{t-1} x_{t-1} + K (y_t - \rho E_{t-1} x_{t-1}), \]
with the steady-state Kalman gain $K = \frac{P}{P + \sigma^2} > 0$ and $P$ is the steady-state variance of the predictive distribution of the latent state.

To model subjective belief, we depart from the above rational learning by assuming that the representative agent put more or less weight in the likelihood function $p(y_t|x_t)$ than a rational agent would. The case with less weight is labeled as “belief in the law of small numbers” by Tversky and Kahneman (1971), where the agent perceives small samples to represent their population equally well as large samples. Such cognitive bias has been widely researched in the literature (see e.g., Rabin, 2002). In this simple state space setting, the posterior distribution is updated using a distorted likelihood function (see, e.g., Santosh, 2021),

$$p(x_t|I_t) \propto p(y_t|x_t)^{1+\theta} \times p(x_t|I_{t-1}),$$  \hspace{1cm} (5)

with $\theta$ capturing the magnitude of cognitive bias. $\theta > 0$ implies that the agent over-reacts to a single observation and $\theta < 0$ implies that the agent under-reacts to a single observation. The subjective belief dynamics follow

$$\tilde{E}_t x_t = \rho \tilde{E}_{t-1} x_{t-1} + \nu (y_t - \rho \tilde{E}_{t-1} x_{t-1}),$$  \hspace{1cm} (6)

where $\tilde{E}(\cdot)$ denotes subjective expectation and the subjective learning gain is $\nu = \frac{(1+\theta)\tilde{P}}{(1+\theta)P + \sigma^2}$.  

To simplify notations, we write the subjective posterior mean $\tilde{E}_t x_t$ as $\tilde{x}_t$ in next subsections.

When $\theta > 0$, the proposed expectation formation has a closely related psychological foundation compared with the diagnostic expectation by e.g., Bordalo et al. (2019, 2020a,b). To see this, we derive the following relation between expectation wedge and

$^9$\tilde{P} is the steady-state variance of the subjective predictive distribution for the latent state, and it may differ from $P$ in the Kalman filter. See their explicit formula in Appendix.

$^{10}$The diagnostic expectation is based on another similar bias, the representativeness heuristic (Kahneman and Tversky, 1972). Santosh (2021) provides more detailed analysis on their difference.
news:

\[ \tilde{E}_{t+1} - E_{t+1} = \frac{\nu - K}{K} (E_{t+1} - E_{t}) + \rho(1 - \nu)(E_{t} - E_{t-1}). \]  \hspace{1cm} (7)

When the subjective learning gain is bigger than the Kalman gain \((\nu > K,)^{11}\) the agent becomes excessively optimistic (pessimistic) after good (bad) news about the latent state, relative to the rational benchmark. It shall be noted that to clarify interpretation, our definition of news refers to the innovations to rational beliefs obtained from the Kalman filter (so that two terms on the right hand side of (7) are uncorrelated). On the other hand, Bordalo et al. (2020a) define the news as innovations to subjective beliefs. Zeng and Zhao (2021) show that these two types of news are highly correlated.

The proposed subjective belief dynamics (6) can be used to parsimoniously capture underreaction to news, when \(\theta\) is small enough. In fact, both over- and under-reactions have been documented empirically in recent macroeconomic and finance literature. For example, Coibion and Gorodnichenko (2015); Bordalo et al. (2020a) find that survey expectations of inflation and GDP growth under-react to news, while earning/dividend growth survey expectations over-react to news (Bordalo et al., 2019, 2020b). Our model-implied inflation and GDP growth expectations are under-reaction to news as in the survey data.

2.2. Subjective expectations on GDP growth and inflation

The four components of GDP – investment spending, net exports, government spending, and consumption – do not move in lockstep with each other. In fact, their levels of volatility greatly differ. Consumption is highly stable and varies less with the business cycle. In contrast, the other three components vary greatly during economic contractions and expansions. For this reason, we assume that there are two latent states for the agent to learn about in the output growth process: a long-run trend growth and a stationary short-run deviation from the trend. The agent learns about long-run trend GDP growth

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^{11}This can be achieved when \(\theta\) is large enough such that \(\theta > \frac{P - \hat{P}}{\hat{P} - \sigma_u^2} \).
only from the stable component (PCE), and they learn about the short-run deviation from the trend by using only the volatile component (GDP growth excluding the PCE). Similarly, for inflation, the agent learns about the long-run trend inflation only from core inflation, and they learn about the stationary short-run deviation from the trend by using only the transitory price changes (the GDP deflator excluding core inflation). Formally, output growth and inflation can be decomposed into the following (account identity):

\[
\Delta g_t = \Delta g^*_{t+1} + \text{Gap}_t^g
\]
\[
\pi_t = \pi^*_{t+1} + \text{Gap}_t^\pi
\]

where \(\Delta g_{t+1}\) and \(\pi_{t+1}\) are the total real GDP growth and inflation, respectively. \(\Delta g^*_{t+1}\) and \(\pi^*_{t+1}\) are growth in real consumption (scaled by total real GDP \(C_{t+1}/GDP_t\)) and core inflation (scaled by total price level \(P^\text{core}_{t+1}/P_t\)), respectively. \(\text{Gap}_t^g\) and \(\text{Gap}_t^\pi\) are the total GDP growth rate excluding \(\Delta g^*_{t+1}\) and the total inflation rate excluding \(\pi^*_{t+1}\), respectively.

The real consumption growth and core inflation follow

\[
\Delta g^*_{t+1} = \mu_{g,t} + \sigma_{g} \varepsilon^*_g_{t+1}
\]
\[
\pi^*_{t+1} = \mu_{\pi,t} + \sigma_{\pi} \varepsilon^*_\pi_{t+1}
\]

where \(\varepsilon^*_g_{t+1}\) and \(\varepsilon^*_\pi_{t+1}\) are i.i.d. standard normal shocks. The latent states are assumed to follow the unit-root processes:

\[
\mu_{g,t+1} = \mu_{g,t} + \sigma_{g} \varepsilon^*_g_{g,t+1}
\]
\[
\mu_{\pi,t+1} = \mu_{\pi,t} + \sigma_{\pi} \varepsilon^*_\pi_{\pi,t+1}
\]

\(^{12}\)To match the yield curve movements in DSGE model, Pflueger and Rinaldi (2020) show the importance of random walk component in inflation expectations.
where $\varepsilon_{g,t+1}^\mu$ and $\varepsilon_{\pi,t+1}^\mu$ are i.i.d. standard normal shocks. The two gap components are assumed to contain stationary latent states:

$$\text{Gap}_i^t = x_{i,t} + \sigma_{i,t}^{gap} \varepsilon_{i,t}^{gap}$$

$$x_{i,t+1} = \rho_i x_{i,t} + \sigma_{i,t}^{x} \varepsilon_{i,t+1}^x,$$

with $i = g, \pi$, and $\varepsilon_{i,t+1}^{gap}, \varepsilon_{i,t+1}^x$ are i.i.d. standard normal shocks.

The agent forms expectations about the latent states based on the same learning scheme in Section 2.1:

$$\tilde{\mu}_{g,t} = \tilde{\mu}_{g,t-1} + v_{g}^*(\Delta g_t^* - \tilde{\mu}_{g,t-1})$$

$$\tilde{\mu}_{\pi,t} = \tilde{\mu}_{\pi,t-1} + v_{\pi}^*(\pi_t^* - \tilde{\mu}_{\pi,t-1})$$

$$\tilde{x}_{g,t} = \rho_g \tilde{x}_{g,t-1} + v_{g}^{gap} (\text{Gap}_g^t - \rho_g \tilde{x}_{g,t-1})$$

$$\tilde{x}_{\pi,t} = \rho_{\pi} \tilde{x}_{\pi,t-1} + v_{\pi}^{gap} (\text{Gap}_\pi^t - \rho_{\pi} \tilde{x}_{\pi,t-1}),$$

where $v_{g}^*, v_{\pi}^*, v_{g}^{gap}$ and $v_{\pi}^{gap}$ are subjective learning gains associated with each latent states. In Section 3, we show that most of them are lower than the corresponding Kalman gains. Thus the agent underreacts to news when updating their forecasts for inflation and real GDP growth, consistent with the findings in Coibion and Gorodnichenko (2015).

### 2.3. Model solutions

Piazzesi and Schneider (2007) show the importance of Epstein and Zin (1989) preferences in generating a sizable inflation risk premium for long-maturity nominal bonds. To illustrate the key role of subjective expectations for bond yield dynamics, we assume that investors have recursive preferences with a CRRA utility function (i.e., they are indifferent between an early or late resolution of uncertainty):

$$V_t(C_t) = \mathbb{E} (U(C_t) + \beta V_{t+1}(C_{t+1})),$$

where $U(C_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma}$, $\gamma$ is the coefficient of the risk aversion and $\beta$ reflects the in-
vestor’s time preference. Note that the agent evaluates the continuation value under their subjective expectations.

2.3.1. Bond pricing

Since the representative agent forms expectations under their subjective beliefs when making portfolio choices, the Euler equation holds under these subjective expectations. Given the CRRA utility function, the log nominal pricing kernel or the nominal stochastic discount factor can be written as follows:

\[ m_{t+1}^s = \log \beta - \gamma \Delta g_{t+1} - \pi_{c,t+1} = \log \beta - v' z_{t+1}, \quad (15) \]

where \( v' = (\gamma, 1) \) and \( z_t = (\Delta g_t, \pi_t)^T \). The time-\( t \) price of a zero-coupon bond that pays one unit of consumption \( n \) periods from now is denoted as \( P_t^{(n)} \) and it satisfies the recursion

\[ P_t^{(n)} = \tilde{E}[M_{t+1}^s | P_{t+1}^{(n-1)}] \quad (16) \]

with the initial condition that \( P_t^{(0)} = 1 \) and \( \tilde{E} \) is the expectation operator under the predictive distribution. Given the fact that different GDP growth and inflation components appear to have very different dynamics in the data, we allow the agent to learn the long-run trend and cyclical components from the data separately. As a result, this model has four state variables: \( \tilde{\mu}_{g,t}, \tilde{\mu}_{\pi,t}, \tilde{x}_{g,t}, \) and \( \tilde{x}_{\pi,t} \). Given the linear Gaussian framework, we assume that \( p_t^{(n)} = \log(P_t^{(n)}) \) is a linear function of these state variables \( \tilde{\mu}_t = (\tilde{\mu}_{g,t}, \tilde{\mu}_{\pi,t})^T \) and \( \tilde{x}_t = (\tilde{x}_{g,t}, \tilde{x}_{\pi,t})^T \):

\[ p_t^{(n)} = -A^{(n)} - B^{(n)} \tilde{x}_t - C^{(n)} \tilde{\mu}_t. \quad (17) \]

When we substitute \( p_t^{(n)} \) and \( p_{t+1}^{(n-1)} \) into Euler equation (16), the coefficients in the pricing equation can be solved with \( C^{(n)} = C^{(n-1)} + v' = v' n, \) \( B^{(n)} = B^{(n-1)} \rho + v' \rho, \) and \( A^{(n)} = A^{(n-1)} + A^{(1)} - 0.5 \text{ Var}_t \left( p_{t+1}^{(n-1)} \right) - \text{ Cov}_t \left( p_{t+1}^{(n-1)}, m_{t,t+1}^s \right) \) (see the appendix.
for details). The log holding period return from buying an $n$ period bond at time $t$ and selling it as an $n-1$ period bond at time $t+1$ is defined as $r_{n,t+1} = p_{t+1}^{(n-1)} - p_t^{(n)}$, and the subjective excess return is $e_{r_{n,t+1}} = -Cov_t \left( r_{n,t+1}, m_{t,t+1}^S \right) = -v'Cov_t \left( z_{t+1}, \tilde{\mu}_{t+1} \right) C^{(n-1)}$. The $n$ period bond yield is defined as $y_t^{(n)} = \log(Y_t^{(n)}) = \frac{1}{n}p_t^{(n)}$.

As we can see from the solution, the yield parameter $(\frac{C^{(n)}}{n})$ on $\tilde{\mu}_t$ is constant over horizon $n$; therefore, the impacts of $\tilde{\mu}_t$ on the long- and short-term yields are the same, which explains the low-frequency movements (trend) in the yields. However, the yield parameter for $x_t$, $\frac{B^{(n)}}{n}$, is decreasing over horizon $n$. Hence, the impact of $x_t$ on the short-term yield is bigger than on the long-term yield, which captures the cyclical movements in the short-term yields. The spread between the long- and short-term yields is mainly driven by the cyclical component $x_t$, which could be positive or negative for many periods (depending on the persistence parameter $\rho$). Still, in contrast with the data, the model-implied spread is mean zero because of the stationarity assumption for $x_t$.

Given the CRRA utility, there is no extra term premium from the agent’s time preference, in contrast to the Epstein and Zin (1989) case. All of the variance and covariance terms (as the term premium in this model) are relatively small in magnitude, which implies a flat yield curve. To solve the price and yields for real bonds, we can simply replace $v'$ with $v' = (\gamma, 0)$.

### 2.3.2. Taylor Rule interpretation

The model-implied one-quarter-ahead nominal yield is a market-based short rate that reflects investors’ subjective expectations (of growth and inflation). Given its performance in matching short-term interest rates in the data (see the next section), one natural question is: what is the connection with the Taylor Rule (Taylor, 1993)? To answer this question, the one-quarter-ahead rate derived from equation (17) can be rewritten as follows:
\[ y_t^{(1)} = A^{(1)} + \gamma \bar{\mu}_g, t + \bar{\mu}_\pi, t + \rho_\pi \bar{x}_\pi, t + \gamma \rho_g \bar{x}_g, t \]
\[ = r^*_t + \bar{\mu}_\pi, t + \rho_\pi \bar{x}_\pi, t + \gamma \rho_g \bar{x}_g, t \]
\[ = i^*_t + \rho_\pi \bar{x}_\pi, t + \gamma \rho_g \bar{x}_g, t, \]

where the neutral real rate of interest \( r^*_t = A^{(1)} + \gamma \bar{\mu}_g, t \) and the neutral nominal rate \( i^*_t = r^*_t + \bar{\mu}_\pi, t \). \( \bar{x}_g, t \) and \( \bar{x}_\pi, t \) are the short-run subjective growth rate and inflation expectations, respectively. The nominal short rate in equation (18) is similar in spirit to a Taylor Rule specification in the literature, with a few exceptions. The first difference is that \( r^*_t \) and \( i^*_t \) move endogenously, instead of as constants. Secondly, \( \bar{x}_g, t \) is the short-run growth-rate deviation from its long-run trend and not an output deviation from its potential in level. Finally, \( \bar{x}_\pi, t \) is the short-run inflation deviation from its long-run trend and not a realized inflation deviation from a constant target.

3. Data, Calibration, and Estimation

The U.S. real GDP growth and the rate of inflation from the GDP deflator are decomposed into one stable component and one transitory component, respectively. We can then calculate the subjective expectations for the long-run trend using PCE and core inflation, and calculate the short-run subjective expectations using data of the transitory components. The model-implied subjective expectations closely match survey-based inflation rate and GDP growth expectations.

3.1. Data

Real output growth, GDP deflator inflation, and their different components are from the Bureau of Economic Analysis for the period 1959.Q2 to 2018.Q2.\(^{13}\) The end-of-quarter yields for 1- to 10-year bonds are from the daily dataset constructed by Gürkaynak et al.

\(^{13}\)Note that the core inflation data are available starting from 1959.Q2.

### 3.2. Parameter estimation and calibration

We assume that the representative agent knows all parameters of state-space models of GDP growth and inflation introduced in Section 2. While we estimate most parameters using maximum likelihood with the Kalman filter, we calibrate the autocorrelation parameter $\rho_g$ that is hard to identify from the data. As widely discussed in the literature (e.g., Bansal and Yaron, 2004; Schorfheide et al., 2018), it is challenging to estimate the persistent component $x_{g,t}$ from the endowment growth series. Therefore, given other parameters, we calibrate $\rho_g$ to maximize the correlation between the model-implied nominal bond yield spread (10-year minus 1-year) and the data counterpart. Panel A of Table 1 provides the parameter values for the state-space models under the physical measure.

The second set of parameters to be determined are the subjective learning gains. Following e.g., Branch and Evans (2006); Cieslak and Povala (2015b), we estimate them by minimizing the root mean square errors (RMSE) between model implied subjective expectations and one-year SPF consensus forecast of real GDP growth and inflation.\(^{14}\) Panel B of Table 1 reports the estimated gains.\(^{15}\) The risk aversion to set to be 3. The time preference $\beta$ is calibrated to match the level of the 1-year nominal yields in the data,

\(^{14}\)To reduce potential to seasonality, we use four quarter average forecasts for GDP growth and inflation.

\(^{15}\)These parameters are within the range of values used in the literature for an optimal Kalman gain. Gilchrist and Saito (2008) use $v = 0.06138$, Edge et al. (2007) use $v = 0.11$. Malmendier and Nagel (2016) show that $v = 0.018$ for the quarterly data that fits the dynamics of the average belief about the inflation expectations in the microdata.
Panel A: parameters of GDP and inflation processes

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_g$</td>
<td>0.33</td>
</tr>
<tr>
<td>$\sigma_{\pi}^\mu$</td>
<td>0.12</td>
</tr>
<tr>
<td>$\sigma_{\pi}^{\text{gap}}$</td>
<td>0.63</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>0.06</td>
</tr>
<tr>
<td>$\rho_{\pi}$</td>
<td>0.935</td>
</tr>
<tr>
<td>$\sigma_{\pi}$</td>
<td>0.08</td>
</tr>
<tr>
<td>$\sigma_{\pi}^{\text{gap}}$</td>
<td>0.09</td>
</tr>
<tr>
<td>$\sigma_{\pi}^x$</td>
<td>0.14</td>
</tr>
<tr>
<td>$\rho_{\pi}$</td>
<td>0.10</td>
</tr>
<tr>
<td>$\rho_{\pi}$</td>
<td>0.93</td>
</tr>
</tbody>
</table>

Panel B: subjective learning gains

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_g^*$</td>
<td>0.013</td>
</tr>
<tr>
<td>$\nu_{\pi}^*$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\nu_{g}^{\text{gap}}$</td>
<td>0.078</td>
</tr>
<tr>
<td>$\nu_{\pi}^{\text{gap}}$</td>
<td>0.232</td>
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Panel C: Kalman gains and other parameters

<table>
<thead>
<tr>
<th>Parameter</th>
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<tbody>
<tr>
<td>$K_g^*$</td>
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<td>$K_{\pi}^*$</td>
<td>0.679</td>
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<tr>
<td>$K_{g}^{\text{gap}}$</td>
<td>0.051</td>
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<tr>
<td>$K_{\pi}^{\text{gap}}$</td>
<td>0.484</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>4</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.0175</td>
</tr>
</tbody>
</table>

Table 1: Model parameters

The table reports model parameter values. Panel A displays parameter estimates for state-space models discussed in Section 2. Panel B reports the subjective learning gains for real GDP growth and inflation, estimated from the corresponding one-year survey data. Panel C reports the risk-aversion coefficient, discount factor, and Kalman gains for the same state-space models. All of the parameters are given in quarterly terms. The standard deviations are in percentages.

which is close to the value in Piazzesi and Schneider (2007). The resulting parameter values are shown in Table 1.

Recent studies show that, relative to the rational expectations, survey forecasts display under- or over-reaction to news, depending on the studied economic series (e.g., Coibion and Gorodnichenko, 2015; Bordalo et al., 2020a). We have shown that our model can generate either under- or over-reaction to observations in data, depending on the relative magnitude of subjective and Kalman gains. Panel B and C of Table 2 report such comparisons for different components of real GDP growth and inflation. Consistent with the empirical evidence, we find that the agent exhibits under-reaction to news for consumption growth, and both the stable and the transitory components in inflation, but over-reaction to news for the transitory component in GDP growth.

To make a formal comparison with Coibion and Gorodnichenko (2015), we run the rationality test for each variable by regressing realized forecast errors on lagged forecast revisions. Intuitively, a positive coefficient on lagged revisions means that the revisions are not enough and the agent still make forecast errors in the same directions as the past shocks, hence, the agent under-reacts to news. Table 2 reports the regression results based on the subjective learning framework discussed in Section 2, and the results include both short horizon and long horizon forecasts. In line with Coibion and Gorodnichenko (2015),
### Table 2: Under- or over-reaction of subjective beliefs

The table reports the rationality test results of model-implied subjective expectations following Coibion and Gorodnichenko (2015). For each variable, the realized forecast errors are regressed on lagged subjective forecast revisions:

\[
x_{t+n} - \tilde{E}_t(x_{t+n}) = \alpha + \beta[\tilde{E}_t x_{t+n} - \tilde{E}_{t-1} x_{t+n}] + \epsilon_{t+n},
\]

with \( n \) chosen to be 1 quarter, 1 year, and 5 years. \( \tilde{E}_t(x_{t+n}) \) denotes subjective forecast implied from agent’s learning discussed in Section 2. The Newey-West t-statistics are reported in parentheses. The sample period is from 1981Q3 to 2018Q2, and 1981Q3 is the quarter when the annual SPF GDP forecasts start to be available.

<table>
<thead>
<tr>
<th>Real GDP growth</th>
<th>Inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1Q</td>
<td>1Y</td>
</tr>
<tr>
<td>( \beta )</td>
<td>3.83</td>
</tr>
<tr>
<td>( (t) )</td>
<td>2.09</td>
</tr>
</tbody>
</table>

The agent under-reacts to news for the short-run real GDP growth, and both short- and long-run inflation, however, the coefficient on long-run GDP growth is not significant.

The interpretation for these findings is potentially related to the Fed’s dual mandates. Private agents know that the Fed is trying to stabilize inflation and unemployment/output growth, and therefore put less weights on shocks to inflation and output growth when updating their expectations.\(^{16}\)

### 4. Empirical findings

Given the analytical solutions and the subjective expectations, we can calculate the model-implied \( r^*_t \), long-, and short-term nominal and real bond yields. We show that the model-implied \( r^*_t \) closely matches the \( r^*_t \) estimates in the literature. As a result, the model-implied 10-year nominal bond yields closely match the historical trend movements in the 10-year Treasury yields. Furthermore, because of the short-run expectations, the model can also explain most of the business cycle movements in short-term Treasury yields and in the yield spreads. Consistent with Duffee (2018), the model-implied nominal bond yield movements are mostly driven by real yield movements, and bond returns are predictable as in Cochrane and Piazzesi (2005).

\(^{16}\)The subjective updating rule agents use can be optimal within a New Keynesian model where the central bank’s goal is to keep the output gap near zero and to keep the inflation rate near its target level.
4.1. Long-run subjective expectations versus $r^*_t$ and $\pi^*_t$

The empirical literature has shown the importance of accounting for macro trends in the term structure of the interest-rate modeling. However, the stationary assumption in the leading equilibrium bond pricing models makes it hard to generate the historically observed low-frequency variations in interest rates. Meanwhile, it is well understood by investors that some components of inflation and GDP growth are more volatile than others. Hence, it is natural for the agent to separately learn the long-run trend and the stationary deviation from the trend. In this paper, agents updates their subjective beliefs of long-run GDP growth and inflation rate using the learning scheme in Section 2, and the subjective expectations follow random-walk processes. Therefore, the model generates low-frequency variations in the posterior beliefs that closely match the $r^*_t$ and $\pi^*_t$ estimates in the literature.

Figure 2 (left panel) shows the model-implied $r^*_t$, which is a linear function of the subjective expectation for the long-run growth trend, $\gamma \tilde{\mu}_{c,t} + Cov$, closely tracks the estimated mean $r^*_t$. The mean $r^*_t$ is taken from Bauer and Rudebusch (2020) and is an average of the three macroeconomic estimates of $r^*$ obtained from Laubach and Williams (2003), Lubik and Matthes (2015), and Kiley (2015). The model-implied $r^*_t$ also closely tracks the trend movements in the 10 year TIPS yields. In fact, because of the the CRRA utility and the stationary assumption for $\tilde{x}_{g,t}$, low-frequency variations in the model-implied 10-year real yield are mainly driven by variations in $\tilde{\mu}_{c,t}$. The correlation between model-implied 10-year real yield and 10-year TIPS is 65% during the period from 2003Q2 to 2018Q2. Figure 2 (right panel) shows that the model-implied $r^*_t$ and the three individual estimates of $r^*$ also closely co-move for most of the sample. The three different $r^*$ estimates diverged from each other before the 1980s. Our model not only captures the low-frequency variations, but its implied $r^*_t$ also exhibits a moderate business cycle component, with dips during recessions and some degree of recovery afterwards.

Figure 3 (left panel) shows that the model-implied subjective expectation of long-run inflation matches very well the trend inflation $\pi^*_t$ obtained from Bauer and Rudebusch (2020). The trend inflation is a survey-based measure, namely, the Federal Reserve’s
The average and individual $r^*$’s (quarterly data) are obtained from Bauer and Rudebusch (2020) for the period 1971:Q4 to 2017:Q2. The three macroeconomic estimates of $r^*$’s are obtained from Laubach and Williams (2003), Lubik and Matthes (2015), and Kiley (2015), respectively. The model-implied $r^*$ (quarterly) covers the period from 1971:Q2 to 2018:Q2. The 10 years TIPS yields are from 2003Q2 to 2018Q2. The gray bars represent periods of recession as defined by the NBER.

The trend inflation $\pi^*$ (quarterly survey-based PTR measure are obtained from FRB/US data) are obtained from Bauer and Rudebusch (2020) from 1971-Q4 to 2017-Q2. The four quarter average of 1-quarter-ahead mean CPI inflation forecasts are from the Philadelphia Fed’s SPF from 1971-Q2 to 2018-Q2. The model-implied inflation expectations of 1-quarter and 5-years are from 1971:Q2 to 2018:Q2. The gray bars represent periods of recession as defined by the NBER.

series on the perceived inflation target rate, denoted as the PTR. It measures the long-run expectations of inflation in the price index of personal consumption expenditures (PCE). Similarly, Figure 3 (right panel) shows that the model-implied subjective expectation for the 1-quarter-ahead inflation also closely tracks the SPF survey for the 1-quarter-ahead inflation.

4.2. Long- and short-term nominal yields, and their spreads

Without trend and cycle decomposition in GDP and inflation, it is difficult for traditional equilibrium models to match both trend and cyclical movements in bond yields. First, while it has long been recognized that nominal interest rates contain a slow-moving trend component (Nelson and Plosser, 1982; Rose, 1988), bond yields in an equilibrium
model (and no-arbitrage term structure models in general) are generally modeled as stationary, mean-reverting processes. As a result, it is hard to explain the low-frequency variations in interest rates in such models, and these are mostly attributed to the term-premium component, which is a residual term in empirical models and is usually interpreted as the inflation risk premium in equilibrium models.

As shown in Figures 2 and 3, the long-run subjective expectations about growth and inflation ($\gamma \tilde{\mu}_{c,t}$ and $\tilde{\mu}_{\pi,t}$) match the macro trends ($r^*_t$ and $\pi^*_t$). An illustration of the potential importance of these beliefs in the 10-year nominal yield is provided in Figure 4 (left panel). The hump-shaped 10-year Treasury yield from the late 1960s to the late 1990s reflects an increase in the inflation expectations before the mid-1980s and a secular decline afterwards. Over the past two decades, as inflation expectations have stabilized, the pronounced decline (especially during the financial crisis periods around 2008) in the expectations for the output growth and, hence, the 10-year real yield, are what mainly drive the down trend in nominal yields. As a result, the model-implied 10-year nominal yield captures the trend movements in the 10-year Treasury yield for the whole sample. Note that the 10-year Treasury yields are higher than the model-implied 10-year nominal yield during the 1970s and 1980s when inflation and GDP growth are negatively correlated. This suggests that a significant amount of inflation risk premium (Piazzesi and Schneider, 2007) might be missed in this model.

Second, learning about the short-run deviations from the trend allows the model to also capture the cyclical movements in the short-term yields. Figure 4 (right panel) shows that the model-implied 1-year nominal yield tracks the historical data for the 1-year Treasury yield relatively well, with one exception. For the post-global financial crisis period, the 1-year Treasury yields were mostly constrained by the zero lower bound, but the model-implied 1-year nominal yields were much more volatile, and these two bond yields recently began to line up again.\footnote{Note that the whole nominal yield curve for the post-global financial crisis period is distorted by the zero lower bound, quantitative easing, and other unconventional monetary policies. The model-implied yield curve for these periods is very different from the data, which can be used as shadow rates, and they started to line up again after 2015.}
The end-of-quarter 10- and 1-year Treasury yields are from Gürkaynak et al. (2007) for the period 1971:Q2 to 2018:Q2. The model implied 10-year real yields, 10-year nominal yields, 1-year nominal yields, and posterior beliefs are obtained for the period 1971:Q2 to 2018:Q2. The gray bars represent periods of recession as defined by the NBER.

In this model, the impacts of $\tilde{\mu}_t$ on the yields of all maturities are the same (capturing the long-run trends). But, as shown in the solution, the impact of $\tilde{x}_t$ (as AR(1) processes), on the short-term yields is bigger than on the long-term yields, which implies that when the short-run beliefs are negative (positive), the spreads would be positive (negative). Hence, the model can generate cyclical movements in the short-term yields and in the spreads, mostly due to variations in $\tilde{x}_t$. Figure 5 (left panel) shows that the model-implied 10-year-minus-1-year yield spreads track the business cycle movements in Treasury yield spreads quite well.

Starting from the previous trough, the short-term nominal yields start to rise when the agent begins to revise their subjective beliefs about the short-run growth and inflation rates upwards towards their long-run trends (the short-run expectations are still negative and the spreads are positive), and the spread starts to shrink as these short-run deviations are turning from negative to positive. This pattern continues until the late expansion stage (or peak) when both the short-run growth rate and inflation expectations are above their long-run means (short-run expectations are positive now), which implies an inverted yield curve. From the previous trough to peak, both the inflation gap and the output gap move from negative to positive, and monetary policy turns from accommodative to contractionary (short-term yields increase), similar to the Taylor Rule. The agent then begins to revise their short-run beliefs sharply downwards, entering a recession (from positive to negative), and the spread switches from negative to positive.
Finally, by comparing the model-implied 1-year nominal yields with the data in Figures 4 (right panel), we observe a recurring pattern (during the 1970s and between 2002 and 2006) that the model-implied short-term yields are higher than the data from the trough to the expansion, and they are lower than those in the data for the late expansion periods. Given that the short-term yields are controlled by the Federal Reserve, this suggests that the Federal Reserve might have been keeping the short rates low for a longer period than suggested by the model (behind the curve) and there was a certain degree of overshoot during the late expansion periods (before recessions). This is consistent with Taylor (2018), who argued that the over-accommodative policy between 2003 and 2005 was a source of the housing bubble.

4.2.1. Quantitative results

As show in the figures above, the model is able to match both the long-run trend and the cyclical variations in the yields. Table 3 shows that the correlation between model-implied yields (yield spreads) and the data are very high (0.83, 0.82, and 0.63 for one-year yields, ten-year yields, and yield spreads respectively), and the volatilities of model implied yields and yield spreads are close to the data. As also shown in the table, the yield autocorrelations are very high in the data, even at five year horizon, which indicates that the yield curve contains a very persistent component. Standard stationary equilibrium models are likely to fail in this dimension because of mean reverting short
rates (because of mean reverting state variables), while this model captures this feature almost perfectly due to the trend plus cycle structure.

However, despite the fact that the model-implied nominal spread can stay positive or negative for an extended period of time at different phases of the business cycle, the level was almost in parallel lower than data (as shown in Figure 5 and Table 3) due to the stationary assumption for $x_t$ and the CRRA utility (hence, the model-implied spread is stationary and mean zero). But, in the data, the spread between the long- and short-term bond yields was positive, in general, and only became inverted before recessions.

The standard equilibrium mechanism used to generate an upward-sloping nominal yield curve is the inflation-risk-premium approach by Piazzesi and Schneider (2007), which relies on inflation as bad news for future growth and the assumption that agents prefer an early resolution to uncertainty. Given that inflation has switched from bad to good news for future growth over the past two decades, conventional equilibrium models would generate negative inflation risk premium and downward-sloping yield curve after the late 1990s. Zhao (2020) assumes that agents are ambiguity-averse with the recursive multiple priors preference of Epstein and Schneider (2003), and provides an alternative approach that can generate both upward-sloping nominal and real yield curves through the term structure of ambiguity. In the appendix, we extend this model by incorporating this worst-case belief approach, and the model-implied short-rate expectations (both nominal and real) are upward-sloping under the agent’s equilibrium worst-case belief and, hence, consistent with the data; the model-implied spread is positive on average.

4.3. Inverted curves, secular stagnation, and recessions

The yield spread movements are mainly drivey by variations in short-run subjective expectations $\tilde{x}_t$. To answer the question why there were more-frequent and deeper inverted curves (accompanied by more-frequent recessions) pre-1990s than post-1990s, we now analyse the historical behaviors of these model-implied short-run expectations.

---

18 The upward-sloping nominal yield curve can also be generated by the real risk premium (Wachter, 2006; Albuquerque et al., 2016; Berrada et al., 2018), and disappointment aversion (Augustin and Tedongap, 2020).
### Table 3: Nominal yields and spreads

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1Y Yield</td>
<td>10Y Yield</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>5.21</td>
<td>6.46</td>
</tr>
<tr>
<td><strong>Std</strong></td>
<td>3.59</td>
<td>2.92</td>
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<tr>
<td><strong>Correlation</strong></td>
<td></td>
<td></td>
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</tbody>
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<table>
<thead>
<tr>
<th></th>
<th>ACC1</th>
<th>ACC2</th>
<th>ACC3</th>
<th>ACC4</th>
<th>ACC12</th>
<th>ACC20</th>
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</thead>
<tbody>
<tr>
<td><strong>Mean</strong></td>
<td>0.97</td>
<td>0.96</td>
<td>0.92</td>
<td>0.89</td>
<td>0.67</td>
<td>0.60</td>
</tr>
<tr>
<td><strong>Std</strong></td>
<td>0.95</td>
<td>0.97</td>
<td>0.95</td>
<td>0.93</td>
<td>0.83</td>
<td>0.82</td>
</tr>
<tr>
<td><strong>Correlation</strong></td>
<td>0.87</td>
<td>0.78</td>
<td>0.71</td>
<td>0.58</td>
<td>-0.14</td>
<td>0.04</td>
</tr>
</tbody>
</table>

This table presents data and model-implied nominal bond yields and their statistics - mean, standard deviation, correlation (between data and model), and autocorrelation (1st-order to 20th-order). Means and standard deviations are annualized percentage rates. The end-of-quarter yields for one- to ten-year Treasury bonds are from the daily dataset constructed by Gürkaynak et al. (2007) from 1972Q2 to 2018Q2.

Figure 5 (right panel) shows that (1) the short-run expectations ($\tilde{x}_{c,t}$ and $\tilde{x}_{\pi,t}$) move in opposite directions before 2000 and they move in the same direction afterwards, and (2) $\tilde{x}_{c,t}$ and $\tilde{x}_{\pi,t}$ were persistently negative for most of the post-2000 period. Given the bond yield solution in Section 2.3, the impacts of $\tilde{x}_{c,t}$ and $\tilde{x}_{\pi,t}$ on short-term nominal yields (and hence on the nominal yield spreads) likely canceled each other out before 2000, and they simultaneously lowered the short-term nominal yields (and hence increased the nominal spreads) afterwards. Therefore, the model can generate an upward shift in the nominal spreads that is consistent with the data in Figure 1. These observations are also consistent with the secular stagnation statement in Summers (2014) that the output gap ($\tilde{x}_{c,t}$ in this model) and the inflation gap ($\tilde{x}_{\pi,t}$ in this model) were persistently negative after 2000. The secular decline in $r_t^*$ and the 10-year Treasury yield were mainly driven by the decline in the the long-run expectations, and the declines in short- to mid-term Treasury yields were further due to the persistently negative $\tilde{x}_{c,t}$ and $\tilde{x}_{\pi,t}$.

---

19 The first observation is consistent with the finding that inflation was bad news for future growth before 2000 (Piazzesi and Schneider, 2007) and switched to good news afterwards (Burkhardt and Hasseltoft, 2012; David and Veronesi, 2013; Campbell et al., 2017; Zhao, 2020).
From a monetary policy point of view, the Federal Reserve faced a trade-off between short-run inflation and growth pre-2000 (the U.S. economy was mostly hit by supply shocks); the Fed had to raise the short-rate to either lower inflation while the output gap was negative (1970-1985) or to lower the output gap while the inflation gap was negative (1985-2000). However, starting in 2000, the U.S. economy was mostly hit by demand shocks, and the Fed could stay low for a longer period of time without facing a trade-off between the inflation and output gaps. Hence, we observed more-frequent recessions for the period before the late 1990s and less-frequent recessions (or longer business cycles) for the period afterwards.

4.4. Yield variance decomposition

Shocks to nominal bond yields consist of news about inflation expectations, expected real short rates, and expected excess returns. Duffee (2018) finds that inflation news account only for 10% to 20% of variances of yield shocks, and leading equilibrium asset pricing models are not consistent with this finding. For example, the long run risk model of Bansal and Yaron (2004) implies a variance ratio of inflation close to 1.

In this model, expected excess returns are constant because of CRRA utility, hence, shocks to nominal bond yields can be decomposed into two parts:

\[ y_{t}^{(n)} - E_{t-1} y_{t}^{(n)} = Cte + \frac{1}{n} \sum_{j=1}^{n} (\tilde{E}_{t} - \tilde{E}_{t-1}) \pi_{t+j} + \frac{1}{n} \sum_{j=1}^{n} (\tilde{E}_{t} - \tilde{E}_{t-1}) r_{t+j-1}, \]

where \( \frac{1}{n} \sum_{j=1}^{n} (\tilde{E}_{t} - \tilde{E}_{t-1}) \pi_{t+j} \) is the shock to inflation expectations and \( \frac{1}{n} \sum_{j=1}^{n} (\tilde{E}_{t} - \tilde{E}_{t-1}) r_{t+j-1} \) captures the news about expected real short rates (or news about real GDP growth expectations in this model). Each part can be further decomposed into trend and cycle components in our model. Table 4 provides a variance decomposition for model-implied 1-quarter, 2-quarter, 1-year, 5-year, and 10-year nominal yields. Consistent with Duffee (2018), the inflation variance ratio is small and real rate news account for most of the variations in nominal yields. Among these different components, the short run growth expectation, \( \tilde{x}_{c,t} \), generates most of the nominal yield variations.
<table>
<thead>
<tr>
<th></th>
<th>1Q</th>
<th>2Q</th>
<th>1Y</th>
<th>5Y</th>
<th>10Y</th>
</tr>
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<tbody>
<tr>
<td><strong>Std of yields shocks</strong></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>8.00</td>
<td>8.49</td>
<td>9.51</td>
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<td>Trend</td>
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<td>0.49</td>
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</tr>
<tr>
<td>Cycle</td>
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<td>7.09</td>
<td>7.92</td>
<td>15.85</td>
<td>25.61</td>
</tr>
<tr>
<td><strong>Inflation variance ratio</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>90.73</td>
<td>90.28</td>
<td>89.36</td>
<td>81.02</td>
<td>71.66</td>
</tr>
<tr>
<td>Trend</td>
<td>1.28</td>
<td>1.38</td>
<td>1.59</td>
<td>4.15</td>
<td>9.18</td>
</tr>
<tr>
<td>Cycle</td>
<td>86.66</td>
<td>86.03</td>
<td>84.70</td>
<td>72.29</td>
<td>56.47</td>
</tr>
<tr>
<td><strong>Real rate variance ratio</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>90.73</td>
<td>90.28</td>
<td>89.36</td>
<td>81.02</td>
<td>71.66</td>
</tr>
<tr>
<td>Trend</td>
<td>1.28</td>
<td>1.38</td>
<td>1.59</td>
<td>4.15</td>
<td>9.18</td>
</tr>
<tr>
<td>Cycle</td>
<td>86.66</td>
<td>86.03</td>
<td>84.70</td>
<td>72.29</td>
<td>56.47</td>
</tr>
<tr>
<td><strong>Cov ratio (real rate, inflation)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.27</td>
<td>1.23</td>
<td>1.13</td>
<td>-0.66</td>
<td>-4.86</td>
</tr>
</tbody>
</table>

This table shows variance decomposition of model-implied 1-quarter, 2-quarter, 1-year, 5-year, and 10-year nominal yields from 1971Q2 to 2018Q2. The standard deviation in first row is in annualized percentage. The variance ratios are percentage of variance coming from different components.

4.5. Bond return predictability

Given the CRRA utility, the subjective bond premium $er_{n,t+1}$ is constant. And the model-implied yields for long-term bonds are roughly equal to the average of the expected future short rates. Thus, the EH holds under the subjective equilibrium belief. However, the economic fundamentals are evolving according to the true data generating processes which are different from agent’s subjective beliefs, hence, the agent makes expectation errors. As a result, the realized bond returns are predictable due to these expectation errors. To formally test the return predictability, we run the single-factor bond premium regressions as in Cochrane and Piazzesi (2005) (CP 2005) with 1 year holding period. The results in Table 5 confirm our intuition that excess bond returns are predictable in this model.

To trace the source of return predictability, the $k$-holding period bond excess return for $n$-period bond can be rewritten as,

$$rx^{(n)}_{t+1:t+k} = \hat{E}_t rx^{(n)}_{t+1:t+k} + (\hat{E}_t i_{t+k} - i_{t+k}) + (\hat{E}_t - \hat{E}_{t+k}) \sum_{j=1}^{n-k-1} i_{t+k+j}$$

where the subjective bond premium $\hat{E}_t rx^{(n)}_{t+1:t+k}$ is constant, $(\hat{E}_t i_{t+k} - i_{t+k})$ is the short-rate errors, and $(\hat{E}_t - \hat{E}_{t+k}) \sum_{j=1}^{n-k-1} i_{t+k+j}$ denotes short-rate revisions. Consistent with
Table 5: Bond excess return predictability

<table>
<thead>
<tr>
<th>Maturity</th>
<th>3Y</th>
<th>4Y</th>
<th>5Y</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Data</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope</td>
<td>0.82</td>
<td>1.20</td>
<td>1.56</td>
</tr>
<tr>
<td>t-stat</td>
<td>4.80</td>
<td>5.11</td>
<td>5.37</td>
</tr>
<tr>
<td>$R^2$</td>
<td>16.92</td>
<td>18.42</td>
<td>19.66</td>
</tr>
<tr>
<td><strong>Model</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope</td>
<td>0.99</td>
<td>1.15</td>
<td>1.22</td>
</tr>
<tr>
<td>t-stat</td>
<td>3.49</td>
<td>3.20</td>
<td>2.92</td>
</tr>
<tr>
<td>$R^2$</td>
<td>9.02</td>
<td>7.55</td>
<td>6.29</td>
</tr>
<tr>
<td><strong>Short-rate errors</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Slope</td>
<td>0.81</td>
<td>0.81</td>
<td>0.81</td>
</tr>
<tr>
<td>t-stat</td>
<td>4.01</td>
<td>4.01</td>
<td>4.01</td>
</tr>
<tr>
<td>$R^2$</td>
<td>11.99</td>
<td>11.99</td>
<td>11.99</td>
</tr>
<tr>
<td><strong>Short-rate revisions</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>t-stat</td>
<td>2.14</td>
<td>2.12</td>
<td>1.87</td>
</tr>
<tr>
<td>$R^2$</td>
<td>3.26</td>
<td>3.22</td>
<td>2.47</td>
</tr>
</tbody>
</table>

This table shows the results for CP 2005 single-factor bond premium regressions (1 year holding period) for 3-, 4-, and 5-year nominal bonds from 1971Q2 to 2018Q2. The model-implied excess bond returns are decomposed into short-rate errors and short-rate revisions. Regression results for each component are provided in this table as well. The end-of-quarter yields for one- to five-year Treasury bonds are from the daily dataset constructed by Gürkaynak et al. (2007) from 1972Q2 to 2018Q2. The model-implied one- to five-year nominal bond yields are obtained for the period 1971:Q2 to 2018:Q2.

Cieslak (2018); Piazzesi et al. (2015); Froot (1989), forecast errors in interest rates are important drivers for bond return predictability. Since the short-rate error for 1-year holding period is the same across different maturity bonds, the coefficients, t-stat, and $R^2$ are the same across maturity. In addition, the short-rate revision is another source of return predictability. In this model, the short-rate errors and revisions are originated from forecast errors and revisions in GDP growth and inflation. This model therefore provides an equilibrium interpretation for the empirical studies that argue expectation errors are the source of bond return predictability (Cieslak, 2018; Piazzesi et al., 2015; Froot, 1989).

5. Robustness

This section provides further checks for the sensitivity of the results in several dimensions.
5.1. Sensitivity of bond yields to macro news

Most equilibrium macro-finance models use final releases of GDP and inflation as inputs, and derive implications for financial markets. However, we know that (i) the final releases are not available to agent in real time and only become available a few quarters/years later, and (ii) financial markets react strongly to macro news in real time, for example, the sensitivity of bond yields to macro news as shown in Gurkaynak et al. (2005) and Swanson and Williams (2014). Therefore, it seems not reasonable to compare these model-implied prices (using final releases as inputs) with real time prices (which are driven by daily news). Specifically for this paper, the model-implied bond yields are calculated using GDP and inflation final releases, which are not directly comparable to real time bond yield data.

However, as shown in Section 4, the model-implied yields match the data very well in this paper. We argue that this is because the real time macro announcements information are embedded in the final releases of GDP and inflation. To show this, we regress total GDP growth/total inflation (and its trend and cycle components as defined in Section 2) on macroeconomic news and monetary policy surprise. Gurkaynak et al. (2005) show that bond yields are sensitive to a set of macroeconomic news and monetary policy surprise. We follow their approach and calculate the surprise/news as the released value less the market expectation from bloomberg. Each macroeconomic news series is scaled by its standard error to make the regression coefficients easily interpretable. The results in Table 6 confirm our argument that GDP/inflation final release (and its trend and cycle components) are predictable by the same set of macroeconomic news and monetary policy surprise that affect real time bond yields. Most of the regressors are statistically significant in predicting at least one of the dependent variables, and hence have significant impacts on the model implied bond yields.

Given these empirical results, we conclude that our model implied bond yields are sensitive to macro news as in Gurkaynak et al. (2005). In general, macro-finance models use final releases of GDP and inflation as inputs are sensitive to real time macro news.
<table>
<thead>
<tr>
<th></th>
<th>$g_{\text{total}}$</th>
<th>$g_{\text{trend}}$</th>
<th>$g_{\text{cycle}}$</th>
<th>$\pi_{\text{total}}$</th>
<th>$\pi_{\text{core}}$</th>
<th>$\pi_{\text{cycle}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Capacity utilization</strong></td>
<td>0.82** (0.30)</td>
<td>0.24 (0.15)</td>
<td>0.57** (0.22)</td>
<td>0.10 (0.13)</td>
<td>-0.09** (0.04)</td>
<td>0.19 (0.13)</td>
</tr>
<tr>
<td><strong>Consumer confidence</strong></td>
<td>0.26 (0.33)</td>
<td>-0.13 (0.17)</td>
<td>0.39 (0.24)</td>
<td>0.01 (0.13)</td>
<td>0.00 (0.04)</td>
<td>0.01 (0.13)</td>
</tr>
<tr>
<td><strong>CPI (core)</strong></td>
<td>-0.56* (0.28)</td>
<td>-0.18 (0.14)</td>
<td>-0.38* (0.21)</td>
<td>-0.07 (0.14)</td>
<td>0.12** (0.05)</td>
<td>-0.19 (0.13)</td>
</tr>
<tr>
<td><strong>Employment cost index</strong></td>
<td>0.28 (0.29)</td>
<td>0.18 (0.14)</td>
<td>0.10 (0.23)</td>
<td>0.02 (0.11)</td>
<td>-0.01 (0.05)</td>
<td>0.03 (0.11)</td>
</tr>
<tr>
<td><strong>GDP (advance)</strong></td>
<td>0.07 (0.26)</td>
<td>0.35** (0.15)</td>
<td>-0.29 (0.19)</td>
<td>-0.21** (0.10)</td>
<td>0.00 (0.05)</td>
<td>-0.21** (0.10)</td>
</tr>
<tr>
<td><strong>Initial claims</strong></td>
<td>-0.25 (0.28)</td>
<td>-0.31** (0.12)</td>
<td>0.05 (0.21)</td>
<td>0.06 (0.12)</td>
<td>-0.13** (0.05)</td>
<td>0.19 (0.12)</td>
</tr>
<tr>
<td><strong>Leading indicators</strong></td>
<td>-0.34 (0.25)</td>
<td>-0.16 (0.13)</td>
<td>-0.18 (0.20)</td>
<td>-0.28** (0.11)</td>
<td>-0.06 (0.05)</td>
<td>-0.22* (0.12)</td>
</tr>
<tr>
<td><strong>NAPM</strong></td>
<td>-0.28 (0.28)</td>
<td>-0.26 (0.15)</td>
<td>-0.02 (0.22)</td>
<td>0.05 (0.14)</td>
<td>0.09 (0.05)</td>
<td>-0.04 (0.12)</td>
</tr>
<tr>
<td><strong>New home sales</strong></td>
<td>0.03 (0.27)</td>
<td>0.36** (0.14)</td>
<td>-0.33 (0.23)</td>
<td>-0.01 (0.15)</td>
<td>0.01 (0.05)</td>
<td>-0.02 (0.15)</td>
</tr>
<tr>
<td><strong>Non-farm payrolls</strong></td>
<td>0.55** (0.23)</td>
<td>0.13 (0.12)</td>
<td>0.42** (0.19)</td>
<td>-0.19* (0.10)</td>
<td>-0.05 (0.03)</td>
<td>-0.14 (0.09)</td>
</tr>
<tr>
<td><strong>PPI (core)</strong></td>
<td>0.12 (0.26)</td>
<td>0.15 (0.15)</td>
<td>-0.03 (0.24)</td>
<td>0.08 (0.12)</td>
<td>0.02 (0.05)</td>
<td>0.06 (0.12)</td>
</tr>
<tr>
<td><strong>Retail sales</strong></td>
<td>0.77*** (0.36)</td>
<td>0.62*** (0.15)</td>
<td>0.15 (0.27)</td>
<td>0.04 (0.14)</td>
<td>0.01 (0.05)</td>
<td>0.02 (0.13)</td>
</tr>
<tr>
<td><strong>Unemployment rate</strong></td>
<td>-0.14 (0.34)</td>
<td>-0.20 (0.14)</td>
<td>0.06 (0.25)</td>
<td>0.02 (0.09)</td>
<td>-0.12** (0.05)</td>
<td>0.14 (0.11)</td>
</tr>
<tr>
<td><strong>Monetary policy surprises</strong></td>
<td>17.22 (10.61)</td>
<td>1.07 (6.30)</td>
<td>16.14* (8.13)</td>
<td>0.80 (6.10)</td>
<td>5.17*** (1.36)</td>
<td>-4.36 (5.81)</td>
</tr>
<tr>
<td><strong>Constant</strong></td>
<td>2.48*** (0.31)</td>
<td>1.67*** (0.16)</td>
<td>0.81*** (0.14)</td>
<td>1.91*** (0.15)</td>
<td>1.12*** (0.05)</td>
<td>0.79*** (0.16)</td>
</tr>
</tbody>
</table>

Robust standard errors in the parentheses. *** indicates significance at the 1-percent level, ** at the 5-percent level, and * at the 10-percent level. The estimated coefficient indicates the response (in annualized percentage rate) of GDP and inflation final releases per standard deviation of the macroeconomic news and surprises in monetary policy announcement. $g_{\text{total}}, g_{\text{trend}}, g_{\text{cycle}}, \pi_{\text{total}}, \pi_{\text{core}}, \pi_{\text{cycle}}$ are total GDP growth, PCE growth, total GDP growth - PCE growth, GDP deflator, core inflation, and GDP deflator - core inflation respectively. All data are quarterly from 1997Q3 to 2017Q3.
6. Conclusion

This paper bridges the gap between empirical and equilibrium yield curve studies by providing an equilibrium interpretation for the trends, cycles, and spreads in historical Treasury bond yields. The representative agent forms subjective expectations about the long-run trend and the short-run deviation from the trend separately using the different components of the GDP growth and inflation rates. The agent puts less weights on data observations than a rational agent would when he/she updates posterior beliefs, which lead to underreact to macro news as in the survey data.

The slow-moving trend component in the yields is driven by the subjective beliefs of the long-run trend inflation and growth rates, which also move closely with the $r_t^*$ and $\pi_t^*$ estimates in the literature. The cyclical movements in the short-term yields and in the spreads between the long- and short-term yields are mostly driven by expectations about the short-run deviation from the trend. The secular stagnation and the upward trend in the Treasury yield spread are tightly coupled because both are driven by persistently negative short-run expectations for both inflation and growth. Consistent with recent empirical findings, short-term nominal yields are mainly driven by variations in short-term real rates (cyclical growth expectations), instead of inflation news. Bond returns are predictable because of the forecast errors and revisions.

Empirical yield curve modeling has been widely used by central banks and practitioners. However, equilibrium yield curve models are rarely used because of their inaccuracy. Given the historical performance of the model and its closed-form solutions, the model can be used for real-time interest-rate forecasting, using survey forecasts or central bank projections for GDP growth and inflation as inputs. To increase its performance, one important future research avenue would be to extend the model and incorporate the inflation risk premium in Piazzesi and Schneider (2007) as well as the potentially stochastic volatility.
References


Appendix

Part I - Model Solution

We solve the benchmark model in part I of the appendix.

A. Expectation formation: rational v.s. subjective

Using the same example as in Section 2.1, rational expectation is defined as Kalman filtering. At steady-state, the posterior (or the filtered) distribution for $x_t$ after observing
$y_t$ is

$$p(x_t|I_t) \propto \exp\left(-\frac{(y_t - x_t)^2}{2\sigma_e^2}\right) \times \exp\left(-\frac{(x_t - \rho E_{t-1}x_{t-1})^2}{2P}\right),$$

with $P$ the steady-state conditional variance of the predictive distribution under the Kalman filter. We then calculate the posterior mean and variance for $x_t$

$$E_t x_t = \rho E_{t-1}x_{t-1} + \frac{P}{P + \sigma_e^2}(y_t - \rho E_{t-1}x_{t-1})$$

$$\text{Var}_t x_t = \frac{\sigma_e^2 P}{P + \sigma_e^2}.$$

Note that by definition, $P$ solves

$$P = \frac{\rho^2 \sigma_e^2 P}{P + \sigma_e^2} + \sigma_u^2.$$

Then we solve for the subjective posterior distribution

$$p(x_t|I_t) \propto \exp\left(-\frac{(1 + \theta)(y_t - x_t)^2}{2\sigma_e^2}\right) \times \exp\left(-\frac{(x_t - \rho \tilde{E}_{t-1}x_{t-1})^2}{2\tilde{P}}\right),$$

with $\tilde{P}$ the steady-state conditional variance of the predictive distribution under the subjective learning. We then calculate the subjective posterior mean and variance for $x_t$

$$\tilde{E}_t x_t = \rho \tilde{E}_{t-1}x_{t-1} + \frac{(1 + \theta)\tilde{P}}{(1 + \theta)\tilde{P} + \sigma_e^2}(y_t - \rho \tilde{E}_{t-1}x_{t-1})$$

$$\tilde{\text{Var}}_t x_t = \frac{\sigma_e^2 \tilde{P}}{(1 + \theta)\tilde{P} + \sigma_e^2}.$$

Note that by definition, $\tilde{P}$ solves

$$\tilde{P} = \frac{\rho^2 \sigma_e^2 \tilde{P}}{(1 + \theta)\tilde{P} + \sigma_e^2} + \sigma_u^2.$$

(1)
In our empirical analysis, we estimate out \( \nu \) from the survey expectation data. Then combining with above definition of \( \tilde{P} \) we can obtain values for \( \theta \) and \( \tilde{P} \).

To derive (7), from the Kalman filter we can write \( y_t \) as

\[
y_t = \frac{E_t x_{t+1} - \rho E_{t-1} x_t}{K} + \rho E_{t-1} x_t = \frac{E_t x_{t+1} - E_{t-1} x_{t+1}}{K} + E_{t-1} x_{t+1},
\]

which is replaced into (6). After subtracting both sides by \( E_t x_{t+1} \), we can obtain (7).

### B. Economic dynamics

The economic dynamics follow

\[
\Delta g_{t+1} = \Delta g_{t+1}^* + \text{Gap}_{t+1}^g \\
\pi_{t+1} = \pi_{t+1}^* + \text{Gap}_{t+1}^\pi,
\]

where \( \Delta g_{t+1} \) and \( \pi_{t+1} \) are the total real GDP growth and inflation, respectively. \( \Delta g_{t+1}^* \) and \( \pi_{t+1}^* \) are the real consumption growth (scaled by total real GDP \( \frac{C_{t+1} - C_t}{GDP_t} \)) and core inflation (scaled by total price level \( \frac{P_{core_{t+1}} - P_{core_t}}{P_{t+1}} \)), respectively. \( \text{Gap}_{t+1}^g \) and \( \text{Gap}_{t+1}^\pi \) are the total GDP growth rate excluding \( \Delta g_{t+1}^* \) and the total inflation rate excluding \( \pi_{t+1}^* \), respectively.

The real consumption growth and core inflation follow

\[
\Delta g_t^* = \mu_g + \sigma_g \varepsilon_{g,t}^* \\
\pi_t^* = \mu_{\pi} + \sigma_{\pi} \varepsilon_{\pi,t}^*,
\]

where \( \varepsilon_{g,t+1}^* \) and \( \varepsilon_{\pi,t+1}^* \) are i.i.d. standard normal shocks. The latent states are assumed to follow the unit-root processes:

\[
\mu_{g,t+1} = \mu_g + \sigma_{g}^\mu \varepsilon_{g,t+1}^\mu \\
\mu_{\pi,t+1} = \mu_{\pi} + \sigma_{\pi}^\mu \varepsilon_{\pi,t+1}^\mu.
\]
where $\varepsilon_{g,t+1}$ and $\varepsilon_{\pi,t+1}$ are i.i.d. standard normal shocks. The two gap components are assumed to contain stationary latent states:

$$\text{Gap}_i^t = x_{i,t} + \sigma^g_{i,t} \varepsilon_{i,t}^g$$
$$x_{i,t+1} = \rho_i x_{i,t} + \sigma^x_{i,t} x_{i,t+1},$$

with $i = g, \pi$, and $\varepsilon_{i,t+1}^{gap}, \varepsilon_{i,t+1}^x$ are i.i.d. standard normal shocks.

The agent forms expectations about the latent states based on the same learning scheme in Section 2.1:

$$\tilde{\mu}_{g,t} = \tilde{\mu}_{g,t-1} + \upsilon_g^* (\Delta g_t^* - \tilde{\mu}_{g,t-1})$$
$$\tilde{\mu}_{\pi,t} = \tilde{\mu}_{\pi,t-1} + \upsilon_{\pi}^* (\pi_t^* - \tilde{\mu}_{\pi,t-1})$$
$$\tilde{x}_{g,t} = \rho_g \tilde{x}_{g,t-1} + \upsilon^g_{gap} (\text{Gap}_g^t - \rho_g \tilde{x}_{g,t-1})$$
$$\tilde{x}_{\pi,t} = \rho_{\pi} \tilde{x}_{\pi,t-1} + \upsilon^\pi_{gap} (\text{Gap}_\pi^t - \rho_{\pi} \tilde{x}_{\pi,t-1}),$$

where $\upsilon_g^*, \upsilon_{\pi}^*, \upsilon^g_{gap}$ and $\upsilon^\pi_{gap}$ are subjective learning gains associated with each latent states.

To solve the model, we first rewrite the dynamics of the whole economy in vector forms as follows:

$$z_{t+1} = \tilde{\mu}_t + \rho_x x_t + \sigma_z^2 \tilde{z}_{t+1}$$
$$x_{t+1} = \rho_x x_t + \upsilon_{gap} (\text{Gap}_t - \rho_x x_t)$$
$$\tilde{\mu}_{t+1} = \tilde{\mu}_t + \upsilon^x (\Delta z_{t+1} - \tilde{\mu}_t)$$

where $z_{t+1} = (\Delta g_{t+1}, \pi_{t+1})^T$, $z_{t+1}^* = (\Delta g_{t+1}^*, \pi_{t+1}^*)^T$, $\text{Gap}_{t+1} = (\text{Gap}^g_{t+1}, \text{Gap}^\pi_{t+1})^T$, $x_{t+1} = (\tilde{x}_{t+1}, \tilde{x}_{t+1}^\pi)^T$, $a_{t+1} = (a_{c,t+1}, a_{\pi,t+1})^T$, $\tilde{\mu}_t = (\tilde{\mu}_{c,t}, \tilde{\mu}_{\pi,t})^T$, $\mu_a = (\mu_a^c, \mu_a^\pi)^T$, $\upsilon_{gap} = \begin{pmatrix} \upsilon_g^{gap} & 0 \\ 0 & \upsilon_{\pi}^{gap} \end{pmatrix}$, $$v^* = \begin{pmatrix} v_c^* \\ v_{\pi}^* \end{pmatrix}, \rho_x = \begin{pmatrix} \rho_c & 0 \\ 0 & \rho_{\pi} \end{pmatrix}, \phi_a = \begin{pmatrix} \phi_a^c & 0 \\ 0 & \phi_a^\pi \end{pmatrix}, \sigma^z = \begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_{\pi} \end{pmatrix},$$
\[ \sigma^a = \begin{pmatrix} \sigma_{ac} & \sigma_{ac}^a \\ \sigma_{a} & \sigma_{a}^a \end{pmatrix}, \quad \tilde{z}_{t+1} = (\tilde{\varepsilon}_{c,t+1}, \tilde{\varepsilon}_{\pi,t+1})^T, \quad \text{and} \quad \varepsilon^a_{t+1} = (\varepsilon_{ac,t+1}, \varepsilon_{a,t+1})^T. \] The shocks \( \tilde{\varepsilon}_{c,t+1}, \tilde{\varepsilon}_{\pi,t+1}, \varepsilon_{d,t+1}, \varepsilon_{ac,t+1}, \) and \( \varepsilon_{a,t+1} \sim i.i.d. N(0, 1). \)

**C. Stochastic discount factor**

Given the CRRA utility, the nominal stochastic discount factor can be written as follows:

\[ m^s_{t,t+1} = \log \beta - \gamma \Delta g_{t+1} - \pi_{c,t+1} = \log \beta - v' z_{t+1}, \]

where \( v' = (\gamma, 1). \) For the real stochastic discount factor, we can replace \( v' \) with \( v' = (\gamma, 0). \)

**D. Bond yields**

The time-\( t \) price of a zero-coupon bond that pays one unit of consumption \( n \) periods from now is denoted as \( P^{(n)}_t \) and it satisfies the recursion,

\[ P^{(n)}_t = \hat{E} [ M^s_{t,t+1} P^{(n-1)}_{t+1} ] \]

with the initial condition that \( P^{(0)}_t = 1 \) and \( \hat{E} \) is the subjective expectation operator. Given the linear Gaussian framework, we assume that \( \tilde{p}^{(n)}_t = \log(P^{(n)}_t) \) is a linear function of \( \tilde{\mu}_t \) and \( x_t \), as follows:

\[ \tilde{p}^{(n)}_t = -A^{(n)} - B^{(n)} x_t - C^{(n)} \tilde{\mu}_t. \]

When we substitute \( \tilde{p}^{(n)}_t \) and \( \tilde{p}^{(n-1)}_{t+1} \) in the Euler equation, the coefficients in the pricing equation can be solved with \( B^{(n)} = B^{(n-1)} + v' \rho, \quad C^{(n)} = C^{(n-1)} + v' = v' n, \) and \( A^{(n)} = \)
\[ A^{(n-1)} + A^{(1)} - 0.5 \times \text{Var}_t \left( p_t^{(n-1)} \right) - \text{Cov}_t \left( p_t^{(n-1)} , m_t^s \right) , \text{ where} \]

\[ \text{Var}_t \left( p_t^{(n-1)} \right) = \left( B^{(n-1)} \right) \text{Var}_t \left( x_{t+1} \right) \left( B^{(n-1)} \right)' + \left( C^{(n-1)} \right) \text{Var}_t \left( \tilde{\mu}_{t+1} \right) \left( C^{(n-1)} \right)' + 2 \left( B^{(n-1)} \right) \text{Cov}_t \left( x_{t+1} , \tilde{\mu}_{t+1} \right) \left( C^{(n-1)} \right)' , \]

\[ \text{Cov}_t \left( p_t^{(n-1)} , m_t^s \right) = v' \text{Cov}_t \left( x_{t+1} , z_{t+1} \right) \left( B^{(n-1)} \right)' + v' \text{Cov}_t \left( \tilde{\mu}_{t+1} , z_{t+1} \right) \left( C^{(n-1)} \right)' , \]

and

\[ A^{(1)} = -\log \beta - 0.5 \times v' \text{Var}_t \left( z_{t+1} \right) v. \]

The nominal bond yields can be calculated as

\[ y_t^{(n)} = -\frac{1}{n} p_t^{(n)} = \frac{A^{(n)}}{n} + \frac{B^{(n)}}{n} x_t + \frac{C^{(n)}}{n} \tilde{\mu}_t. \]

The log holding period return from buying an \( n \) periods bond at time \( t \) and selling it as an \( n - 1 \) periods bond at time \( t - 1 \) is defined as \( r_{n,t+1} = p_{t+1}^{(n-1)} - p_t^{(n)} \), and the subjective excess return is \( er_{n,t+1} = -\text{Cov}_t \left( r_{n,t+1} , m_t^s \right) = -\text{Cov}_t \left( p_{t+1}^{(n-1)} , m_t^s \right) \). To solve the price and yields for the real bonds, we can simply replace \( v' \) with \( v' = (\gamma , 0) \).

## Part II - Model with Ambiguity

In this extended model, we continue to consider an endowment economy with a representative agent who has a CRRA utility function. However, the agent is assumed to have limited information about the stochastic environment and, hence, faces both risk and ambiguity. Here, the risk refers to the situation where there is a probability law that guides the choice. At the same time, the ambiguity-averse agent (with recursive multiple priors or a maxmin preference by Epstein and Schneider 2003) lacks the confidence to
assign probabilities to all of the relevant events. Instead, they act as if they are evaluating future prospects using a worst-case probability drawn from a set of multiple distributions.

Investors in this economy have in mind a benchmark or reference measure of the economy’s dynamics that represents the best estimate of the stochastic process. The reference measure is the full stochastic environment in the benchmark model (including the subjective expectations). But the agent is concerned that the reference measure is misspecified and believes that the true measure is actually within a set of alternative measures that are statistically close to the reference distribution. Equilibrium prices adjust such that the ambiguity-averse agent is happy to consume the output as an endowment.

A. Ambiguity about economic dynamics

We assume the total predictive distribution from the benchmark model as the reference measure, which can be rewritten as follows:

\[
\begin{align*}
\Delta g_{t+1} &= \tilde{\mu}^*_c + \rho_c \tilde{x}_{c,t} + \varepsilon_{c,t+1} \\
\pi_{t+1} &= \tilde{\mu}^*_\pi + \rho_\pi \tilde{x}_{\pi,t} + \varepsilon_{\pi,t+1},
\end{align*}
\]

where \( \varepsilon_{c,t+1} \) is a combination of \( \tilde{\varepsilon}^*_{c,t+1}, \tilde{\varepsilon}^\text{gap}_{c,t+1}, \) and \( \tilde{\varepsilon}^x_{c,t+1} \); \( \varepsilon_{\pi,t+1} \) is a combination of \( \tilde{\varepsilon}^*_{\pi,t+1}, \tilde{\varepsilon}^\text{gap}_{\pi,t+1}, \) and \( \tilde{\varepsilon}^x_{\pi,t+1} \). \( \tilde{\mu}^*_c, \tilde{\mu}^*_\pi, \tilde{x}_{c,t}, \) and \( \tilde{x}_{\pi,t} \) have exactly the same dynamics as in the benchmark model. However, the agent is concerned that their reference measure is misspecified and that the true measure is actually within a set of alternative measures that are statistically close to the reference measure. The set of alternative measures is generated by a set of different mean output growth (inflation) rates around the reference mean value \( \tilde{\mu}^*_c + \rho_c \tilde{x}_{c,t} (\tilde{\mu}^*_\pi + \rho_\pi \tilde{x}_{\pi,t}) \). Specifically, under alternative measure \( p^\text{a} \), the output growth and inflation rates are as follows:

\[
\begin{align*}
\Delta g_{t+1} &= \tilde{a}_{c,t} + \tilde{\mu}^*_c + \rho_c \tilde{x}_{c,t} + \tilde{\varepsilon}_{c,t+1} \\
\pi_{t+1} &= \tilde{a}_{\pi,t} + \tilde{\mu}^*_\pi + \rho_\pi \tilde{x}_{\pi,t} + \tilde{\varepsilon}_{\pi,t+1},
\end{align*}
\]
where $\tilde{a}_{c,t} \in A_{c,t} = [\tilde{\mu}_{c,t} + \rho_c \tilde{x}_{c,t} - a_{c,t}, \tilde{\mu}_{c,t} + \rho_c \tilde{x}_{c,t} + a_{c,t}]$ and $\tilde{a}_{\pi,t} \in A_{\pi,t} = [\tilde{\mu}_{\pi,t} + \rho_\pi \tilde{x}_{\pi,t} - a_{\pi,t}, \tilde{\mu}_{\pi,t} + \rho_\pi \tilde{x}_{\pi,t} + a_{\pi,t}]$ with both $a_{c,t}$ and $a_{\pi,t}$ being positive. Each trajectory of $\tilde{a}_t$ will yield an alternative measure $\hat{p}$ for the joint process. A larger $a_{c,t}(a_{\pi,t})$ implies that investors are less confident about the reference distribution.\textsuperscript{20}

Using the forecast dispersion from the Blue Chip Financial Forecast (BCFF) survey as a measure for the size of the ambiguity, Zhao (2020) finds that, before the late 1990s, the size of the ambiguity for long-horizon inflation was bigger than that for short horizons and that this pattern was reversed afterwards. However, the size of the ambiguity for long-term real output growth was always smaller than those for the short term. Together with the fact that the positive inflation shocks changed from negative to positive news about future growth in the past 20 years, the ambiguity-averse agent chose the upper bound ($a_{\pi,t}$) as the worst-case measure for inflation before the late 1990s and the lower bound ($-a_{\pi,t}$) as the worst-case measure afterwards, while for output growth (as the endowment), the agent always chose the lower bound ($-a_{c,t}$) as the worst-case measure in equilibrium. Zhao (2020) showed that the expectations hypothesis roughly holds under investors’ worst-case beliefs and the upward-sloping nominal and real yield curves are mainly driven by the upward-sloping nominal and real short-rate expectations.

We follow Zhao (2020) and model $a_{c,t}$ ($a_{\pi,t}$) as a random walk with drift as follows:

\[
\begin{align*}
a_{c,t+1} &= \mu_c^a + a_{c,t} + \sigma_{ac} \varepsilon_{ac,t+1} + \sigma_{ac}^a \varepsilon_{a,t+1} \\
a_{\pi,t+1} &= \mu_\pi^a + a_{\pi,t} + \sigma_{a\pi}^a \varepsilon_{a,t+1},
\end{align*}
\]

where $\mu_c^a$ and $\mu_\pi^a$ are the drift parameters, which can be positive or negative. $a_{c,t}$ and $a_{\pi,t}$ are driven by a common exogenous shock $\varepsilon_{a,t+1}$, where the coefficients $\sigma_{ac}^a$ and $\sigma_{a\pi}^a$ capture the correlation between them. $\varepsilon_{ac,t+1}$ is an $a_{c,t}$ specific shock that captures the difference of these two.

\textsuperscript{20}Ilut and Schneider (2014) link the size of ambiguity with the observed volatility under the reference measure and provide a detailed discussion for the source of ambiguity.
B. Preference: recursive multiple priors

To illustrate the key role of the ambiguity yields, we assume investors have a recursive multiple-priors preference axiomatized by Epstein and Schneider (2003) but with the same CRRA utility function:

\[ V_t(C_t) = \min_{p_t \in P_t} \mathbb{E}_{p_t} \left( U(C_t) + \beta V_{t+1}(C_{t+1}) \right), \]

where \( U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma} \), \( \gamma \) is the coefficient of the risk aversion and \( \beta \) reflects the investor’s time preference.

C. Bond pricing

The Euler equation holds under the representative agent’s worst-case belief, and the log nominal pricing kernel is the same as in the benchmark model. The time-\( t \) price of a zero-coupon bond satisfies the same recursion, but the expectation is under the worst-case belief.

The benchmark model has four state variables (the posterior beliefs \( \tilde{\mu}_{c,t}, \tilde{\mu}_{\pi,t}, \tilde{x}_{c,t}, \) and \( \tilde{x}_{\pi,t} \)) that explain the trends and cycles in the yields. To generate upward-sloping nominal and real yield curves, we add two more state variables in this model: \( a_{c,t} \) and \( a_{\pi,t} \). Given the linear Gaussian framework, we assume that \( p_t^{(n)} = \log(P_t^{(n)}) \) is a linear function of these state variables \( \tilde{\mu}_t^* = (\tilde{\mu}_{c,t}, \tilde{\mu}_{\pi,t})^T, x_t = (\tilde{x}_{c,t}, \tilde{x}_{\pi,t})^T \) and \( a_t = (a_{c,t}, a_{\pi,t})^T \):

\[ p_t^{(n)} = -A^{(n)} - B^{(n)} x_t - C^{(n)} \tilde{\mu}_t^* - D^{(n)} a_t. \]  

When we substitute \( p_t^{(n)} \) and \( p_{t+1}^{(n-1)} \) in the Euler equation (10), the coefficients in the pricing equation can be solved with \( B^{(n)} = B^{(n-1)} + v' \phi_a = v'n\phi_a, \) and \( A^{(n)} = A^{(n-1)} + A^{(1)} - 0.5 \) \( \text{Var}(p_t^{(n-1)}) - \text{Cov}(p_t^{(n-1)}, m_{t+1}^{(n)}) + D^{(n-1)} \mu_a, \) where \( \phi_a \) represents the equilibrium choice of the upper or lower bounds, which are equal to \(-1\) or \(+1\) on the diagonal. All of the variance and covariance terms are
relatively small in the data. Hence, given the CRRA utility, the subjective excess return is small in this model.

As in the benchmark model, $\tilde{\mu}_t^*$ explains the low-frequency movements (trend) in the yields and $x_t$ captures the cyclical movements in the short-term yields. The yield parameter $\frac{D(n)}{n}$ for ambiguity $a_t$ is $v'\phi_a$ (constant over horizon $n$); hence $a_{\epsilon,t}$ lowers the yields for the whole sample period ($\phi^a_{\epsilon} = -1$), and $a_{\pi,t}$ lowers (raises) the yields for the second subperiod when $\phi^a_{\pi} = -1$ (the first subperiod when $\phi^a_{\pi} = 1$). The impacts of $a_t$ on the long- and short-term yields are the same, and the upward-sloping nominal and real yield curves are mainly driven by increases in $\frac{D(n)}{n}$ over horizon $n$ due to $\mu_a$. To solve the price and yields for real bonds, we can simply replace $v'$ with $v' = (\gamma, 0)$.

D. Model intuition

Given the closed-form solution, the intuition of the model follows directly from the fact that interest rates reflect investors’ worst-case expectations.

First, the agent chooses the lowest growth rate as their worst-case belief in equilibrium; thus, the ambiguity about growth pushes down real yields. Given that the size of the ambiguity for growth is higher for short horizons, short-term real rates are pushed down by more than long-term real rates. Therefore, the real yield curve slopes upward.

Second, the ambiguity about the inflation rate contributes to an upward-sloping nominal yield curve but for different reasons in the two regimes. Pre-2000, positive inflation shocks were bad news for future growth – the worst-case inflation was the highest rate; thus, the ambiguity about inflation pushed up nominal yields. Since there was more ambiguity about long-run inflation, the long-term nominal yields were pushed up by more than just the short-term nominal yields. Post-2000, positive inflation shocks were good news for future growth – the worst-case inflation was the lowest rate; thus, the ambiguity about inflation pushed down the nominal yields. But now there is more ambiguity about the short-run inflation, and the short-term nominal yields are pushed down farther than the long-term nominal yields are. In both cases, the model implies an upward-sloping and steeper nominal yield curve.
E. Empirical findings

We use the same data sets as in the benchmark model. We also use the forecast dispersions for the real output growth and inflation obtained from the Philadelphia Fed’s SPF as a measure for the realized size of the ambiguity. Then we can calculate the realized values for all of the state variables and, hence, the model-implied yields. Since the only change in this extended model is the ambiguity, the model can still match the historical trends and cycles in the yields as in the benchmark model. In addition, the model generates upward-sloping nominal and real yield curves as in the data.

E.1. Parameters

All of the parameters (excluding the ambiguity parameters and $\beta$) are the same as in the benchmark model. The ambiguity parameters are the same as in Zhao (2020), where the whole sample period is split into two subperiods and the parameters are different for each subperiod (mainly the trend parameter for the inflation ambiguity $\mu_\pi^a$). The realized size of the ambiguity is measured by the past one-year average of the SPF forecast dispersions that are calculated by the 60th percentile minus the 40th percentile of the individual forecasts. $\beta$ is calibrated to match mean of 1-year nominal yield with the data.

E.2. Trends, cycles, and spreads in the yields

Since the parameters for the inflation and GDP are exactly the same as in the benchmark model; the posterior beliefs for the short- and long-term inflation expectations in this model are also exactly the same as shown in Figures 3, and the model-implied $r^*_t$ is almost the same as before.\footnote{To be consistent with the concept of $r^*$, neither the short-run effect from $\tilde{x}_{ct}$ on the real yield, nor the effect of ambiguity on the yields, is included for calculation of the model-implied $r^*$.} The model-implied short- and long-term yields can also match the data well, and graphs of nominal yields are provided in an earlier version of this paper. We will skip these graphs in this version.

The limitation of the benchmark model is that the model-implied spread is mean zero because of the stationarity assumption for $x_t$. The short-rate expectations (both
nominal and real) are upward-sloping under the agent’s equilibrium worst-case belief in this model and, hence, are consistent with the data. The model-implied spread is positive, on average. Figure 6 shows that the dynamics of the model-implied spread match the data well.

The end-of-quarter 10-year minus 1-year nominal yield spreads are obtained from Gürkaynak et al. (2007) for the period from 1968:Q3 to 2018:Q2. The model-implied 10-year minus 1-year nominal yield spread represents the period from 1968:Q3 to 2008:Q4. The gray bars represent periods of recession as defined by the NBER.

E.3. Expectations hypothesis and predictability of bond returns

Investors’ subjective nominal and real short-rate expectations, reflecting their worst-case beliefs in growth and inflation, are upward sloping. And given the CRRA utility (the subjective bond premium $er_{n,t+1}$ is close to zero), the model-implied yields for long-term bonds are roughly equal to the average of the expected (worst-case) future short rates. Thus, the EH roughly holds under the subjective equilibrium belief. However, because long-run growth and inflation evolve over time under a true distribution that is different from their worst-case beliefs, investors’ ambiguity about long-run inflation or GDP growth does not materialize when the time arrives. At each time $t$, the realized one-step-ahead ambiguity ($a_{1c,t}$ or $a_{1\pi,t}$) contains only the random walk with no trend (the trend has not materialized). Hence, the realized one-step-ahead ambiguity does
not become larger or smaller as investors had perceived in the past, and the realized short rates (nominal and real) are lower than expected under their worst-case beliefs. These differences and the current yield spreads/forward rates are both driven by a trend component in the ambiguity process. Hence, consistent with the empirical evidence, the realized excess bond returns are predictable. To an observer outside the model, the difference between the worst-case expectation and the realized short rate (governed by the benchmark measure) looks like expectational error but is rational for agents inside the model who face ambiguity, thus, providing a rational interpretation for the expectational errors in Froot (1989), Piazzesi et al. (2015), and Cieslak (2018). A detailed discussion and formal tests of the EH (Campbell and Shiller, 1991; Cochrane and Piazzesi, 2005) are provided in Zhao (2020).