Kink-like Solutions for the FPUT Lattice and the mKdV as a Modulation Equation

Trevor Norton

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The FPUT System

The FPUT lattice

- The Fermi-Pasta-Ulam-Tsingou (FPUT) lattice is an infinite set of differential equations posed on $\mathbb{Z}$

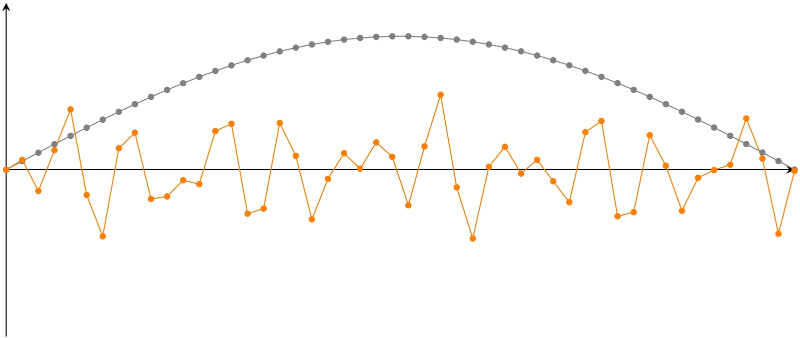
$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z} \tag{FPUT}$$

where $V(x)$ is the potential. Alternatively, it can be written in the strain variables $u_n = x_n - x_{n-1}$

$$\ddot{u}_n = V'(u_{n+1}) - 2V'(u_n) + V'(u_{n-1}), \quad n \in \mathbb{Z}.$$

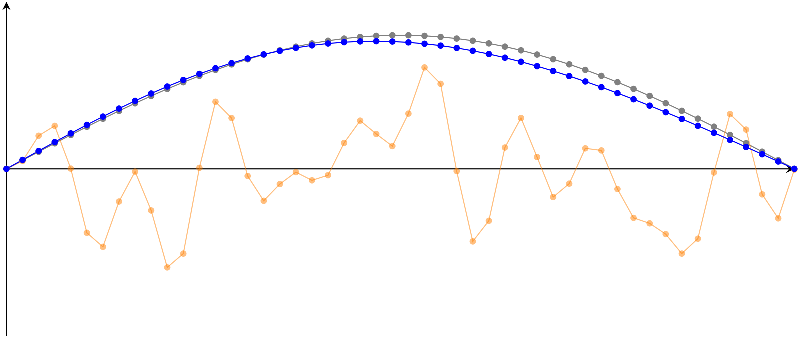
- Modeled the thermalization of solid.
- Researchers numerically computed solutions of the FPUT on a large, finite grid with potential $V(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3$ and got a surprising result.

Starting with energy concentrated in lowest mode:



In the short-term, we see the expected behavior.

Starting with energy concentrated in lowest mode:



Eventually, the system has a near-recurrence of initial condition ($\approx 97\%$ of energy returns to lowest mode)!

The KdV as a modulation equation

- It was shown in [ZK65] that the Korteweg-de Vries (KdV) equation (after a rescaling) is a continuum limit for the FPUT with potential $V(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$

$$u_t - 6uu_x + u_{xxx} = 0 \tag{KdV}$$

- The KdV has soliton solutions

$$-\frac{c}{2}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)$$

- Solitons can pass through each other without changing shape. This helps explain the recurrence.

Some results

- Solitary wave solution of FPUT has profile that can be approximated by the profile of the soliton solution. Solitary wave solution is stable [FP99, FP02, FP03, FP04].
- Asymptotic stability in front of the solitary wave in ℓ^2 [Miz09]. Later expanded to N-solitary wave solutions [Miz13].
- KdV can be used to approximate small-amplitude, long-wavelength solutions [SW00, KP17].

The β-FPUT chain

- When $V(x) = \frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots$ the continuum limit for the FPUT is given now by the defocusing modified KdV (mKdV)

$$u_t - 6u^2 u_x + u_{xxx} = 0 \tag{mKdV}$$

  which has kink solutions given by

$$\varphi_c(x + ct) = \sqrt{\frac{c}{2}} \tanh\left(\sqrt{\frac{c}{2}}(x + ct)\right).$$

- In [PRC19] the recurrence was studied numerically, and it seems to be driven in part by kink-like solutions to the FPUT.
- There are few rigorous results explaining the relationship between the FPUT with this potential and the defocusing mKdV and its kinks.

Research Topics

1. Show that there exists a traveling wave solution whose profile can be approximated using the kink solution of the defocusing mKdV.

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The problems of existence and stability are common in the study of traveling wave solutions of PDEs.

Existence of the Kink-Like Solution

Strategies to show existence

- [FP99] set up a fixed point argument to get the profile of the solitary traveling wave.
    - Took a Fourier transform
    - Defined a map from $H^1(\mathbb{R})$ to itself where the fixed point is the profile of the solitary wave solution
- This is difficult to replicate for the kink-like solution since its profile cannot lie in $H^1(\mathbb{R})$.
- In [Ioo00], a center manifold reduction to show the existence of traveling wave solutions of the FPUT.
    - The existence of the solution we are interested is proved, but its profile was not estimated.
    - Want to show that the profile is close to φ_1 under re-scaling.

Outline of argument

1. Follow the construction of the center manifold given by Iooss, while carefully keeping track of the explicit terms.
2. Show that (under a change of coordinates) that as we send $c \nearrow 1$ the profile will be exactly φ_1.
3. Apply Fenichel theory to show that when $c < 1$, the profile of the kink-like solution stays near φ_1.

Traveling wave ansatz

- We make the assumption that the solution is a traveling wave with speed $c > 0$;

$$x_n(\tilde{t}) = x(n - c\tilde{t})$$

  so that $x(t)$ satisfies the advance-delay differential equation

$$\ddot{x}(t) = \mu\Big(V'(x(t+1) - x(t)) - V'(x(t) - x(t-1))\Big)$$

  where $\mu = c^{-2}$.
- Want to convert this into a semi-dynamical system.
- Typically, for second order differential equations, we can take $\xi(t) = \dot{x}(t)$ and rewrite the equation as a first order system.
- To deal with the delay terms, we take a "slice" of the function $x(t)$ from $[t-1, t+1]$: $X(t, v)$ where $\nu \in [-1, 1]$.

Abstract ODE

- Let $U(t) = (x(t), \xi(t), X(t, \cdot))$ so that the advance-delay equation becomes

$$\partial_t U = L_\mu U + M_\mu(U)$$

where

$$L_\mu = \begin{pmatrix} 0 & 1 & 0 \\ -2\mu & 0 & \mu(\delta^1 + \delta^{-1}) \\ 0 & 0 & \partial_v \end{pmatrix}$$

and

$$M_\mu(U) = \mu(0, g(\delta^1 X - x) - g(x - \delta^{-1} X), 0)^T$$

- Second component is the advance-delay equation and third component $\partial_t X = \partial_v X$ just shifts the “slice” $X(t, \cdot)$ forward in time.

Center manifold

- Iooss shows that when $\mu = \mu_0 = 1$, L_{μ_0} has a quadruple zero eigenvalue and the rest of the spectrum is uniformly bounded away from $i\mathbb{R}$.
- Thus we can find a four-dimensional center manifold parameterized by μ.
- One of our coordinates partially decouples from the system, so we can reduce the original system by one dimension and get a three-dimensional center manifold instead.
- Solutions on the center manifold of our reduced system are given by

$$W = A\zeta_1 + B\zeta_2 + C\zeta_3 + \Phi_\mu(A, B, C)$$

Computing dynamics on the center manifold

- Using the facts that (1) the center manifold is invariant and (2) the image of Φ_μ is orthogonal to the generalized eigenvectors, we can compute terms in the Taylor series of Φ_μ and the equations of motion for A, B, C
- We still get something messy.
- Idea: we are looking for small-amplitude, long-wavelength solutions, so we will introduce a smallness parameter ϵ and make a change of variables, neglecting higher orders of ϵ
  - $c^2 = 1 - \epsilon^2/12$
  - $A(t) = \epsilon \underline{A}(\epsilon t),\ B(t) = \epsilon^2 \underline{B}(\epsilon t),\ C(t) = \epsilon^3 \underline{C}(\epsilon t).$

Computing dynamics on the center manifold

- Then the equations of motion on the center manifold become

$$\begin{aligned}\underline{A}' &= \underline{B} + \mathcal{O}(\epsilon^2)\\ \underline{B}' &= \underline{C}\\ \underline{C}' &= -\underline{B} + 6\underline{A}^2\underline{B} - 2\epsilon V^{(5)}(0)\cdot \underline{A}^3\underline{B} + \mathcal{O}(\epsilon^2),\end{aligned}$$

  where $'$ is the derivative with respect to $s = \epsilon t$.
- Formally setting $\epsilon = 0$ gives a system equivalent to

$$\underline{A}''' + \underline{A}' - 6\underline{A}^2\underline{A}' = 0$$

  which has $\underline{A}(s) = \varphi_1(s)$ as a solution.
- This corresponds to a heteroclinic orbit on the center manifold. Showing that this persists for $\epsilon > 0$ will give us the estimate we want.

Persistence of heteroclinic orbit

- To show that the heteroclinic orbit persists, we show that it lies on the intersection of a stable and unstable manifold that intersect transversally for $\epsilon = 0$.
- Define the manifolds

$$\overline{M} = \{(\underline{A}, 0, 0, \epsilon) \in \mathbb{R}^4 : |(\underline{A} + 1/\sqrt{2}, \epsilon)| \leq \delta\}$$
$$\overline{N} = \{(\underline{A}, 0, 0, \epsilon) \in \mathbb{R}^4 : |(\underline{A} - 1/\sqrt{2}, \epsilon)| \leq \delta\}$$

  for $\delta > 0$ small enough.
- The flow is hyperbolic in the normal directions on the manifolds, and so $\overline{M}$ has an unstable manifold $\mathcal{M}_\epsilon$ and $\overline{N}$ has a stable manifold $\mathcal{N}_\epsilon$, both continuously parameterized by ϵ.
- We want to show that a transverse intersection occurs at $\epsilon = 0$. We have the exact dynamics in this case, and can find $\mathcal{M}_0$ and $\mathcal{N}_0$ explicitly.

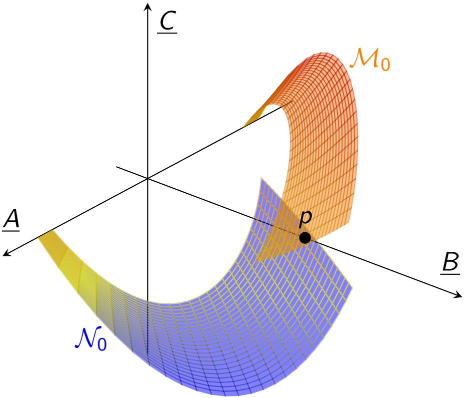

Figure: The above figure shows the stable and unstable manifolds for $\epsilon = 0$. The manifolds $\mathcal{M}_0$ and $\mathcal{N}_0$ have a transverse intersection at $p = (0, 1/2, 0)$.

Since the stable and unstable manifolds vary continuously with respect to ϵ, this will give us a result. However, we can strengthen the result by assuming more about $V(x)$:

$$\begin{aligned} &\text{(H1)} \quad V(x) = \tfrac{1}{2}x^2 - \tfrac{1}{24}x^4 + \mathcal{O}(x^5) \\ &\text{(H2)} \quad V(x) = \tfrac{1}{2}x^2 - \tfrac{1}{24}x^4 + \mathcal{O}(x^6) \\ &\text{(H3)} \quad V(x) = \tfrac{1}{2}x^2 - \tfrac{1}{24}x^4 \end{aligned}$$

- If (H1) holds, then the profile will be ϵ close to φ_1.
- If (H2) holds , then the manifolds vary continuously with respect to ϵ^2 and we can improve the estimate to order ϵ^2.
- If (H3) holds, then the asymptotic limits of the solution on the center manifold will be the same as φ_1 and the two will differ by an H^1 function.

Theorem 1

There exists $\epsilon_0 > 0$ and $C > 0$ such that for every $\epsilon > (0, \epsilon_0]$ there is a traveling wave solution given by $u_n(t) = u_c(n - ct)$ with positive wave speed $c^2 = 1 - \epsilon^2/12$. Furthermore, we have the additional estimates on the wave profile of u_c.

- *If (H1) holds, then*
$$\left\| \frac{1}{\epsilon} u_c \left(\frac{\cdot}{\epsilon} \right) - \varphi_1 \right\|_{C^3} \leq C\epsilon$$
- *If (H2) holds, then*
$$\left\| \frac{1}{\epsilon} u_c \left(\frac{\cdot}{\epsilon} \right) - \varphi_1 \right\|_{C^3} \leq C\epsilon^2$$
- *If (H3) holds, then*
$$\left\| \frac{1}{\epsilon} u_c \left(\frac{\cdot}{\epsilon} \right) - \varphi_1 \right\|_{H^3} \leq C\epsilon$$

Long-time approximations of FPUT solutions

Long-time approximations

- In [SW00], it is shown that the KdV serves as a modulation equation for counter-propagating wave solutions of the FPUT with potential $V(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$. The approximation holds for small-amplitude, long-wavelength solutions for ϵ^{-3} time.
- A similar result is shown in [KP17] for a single traveling wave solution, but for time scales of order $\epsilon^{-3}\log|\epsilon|$. This allows one to show meta-stability of solitary wave solutions from orbital stability of the KdV solitary waves.
- Traveling wave ansatz: assume solution of section 1 with potential $V(x) = \frac{1}{2}x^2 - \frac{1}{24}x^4$ can be expressed as

$$u_n(t) \approx \epsilon f(\epsilon(n+t), \epsilon^3 t) + \epsilon g(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \phi(\epsilon n, \epsilon t)$$

  where f has fixed non-zero limits $f_{\pm\infty}$ at positive and negative infinity and

$$c = c(\epsilon, f_\infty) = 1 - \frac{\epsilon^2 f_\infty^2}{4}$$

Modulation equations

Plugging in the ansatz, we get that the approximation holds formally up to ϵ^5 if f, g, ϕ satisfy

$$2\partial_2 f = -\frac{1}{6}\partial_1(f^3) + \frac{1}{12}\partial_1^3 f$$

$$-2\partial_2 g = -\frac{1}{6}\partial_1(g^3 + 3f_\infty g^2) + \frac{1}{12}\partial_1^3 g,$$

$$\begin{aligned}\partial_2^2\phi(\xi,\tau) = \partial_1^2\phi(\xi,\tau) - \frac{1}{6}\partial_1^2\big[&3(f^2(\xi+\tau,\epsilon^2\tau) - f_\infty^2)g(\xi - c\tau,\epsilon^2\tau) \\ &+ 3(f(\xi+\tau,\epsilon^2\tau) - f_\infty)g^2(\xi - c\tau,\epsilon^2\tau)\big]\\ \phi(\xi,0) = \partial_1\phi(\xi,0) = 0.&\end{aligned}$$

Proof strategy

1. Show that the interference term remains uniformly bounded on the time scale
    - Show $(f - f_\infty) \cdot g$ decays rapidly in time
2. Show that the error remains of order ϵ^3 on the time scale
    - Usually done by choosing an appropriate energy function to bound the error terms

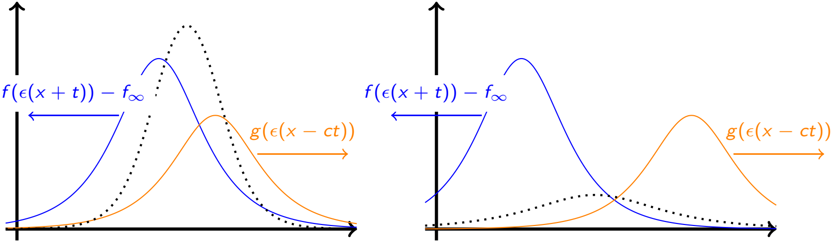

Figure: The function $f(\epsilon(x+t)) - f_\infty$ (shown in blue) moves to the left while $g(\epsilon(x-ct))$ (shown in orange) moves to the right. Since they are localized, the product (shown by the dotted line) will quickly decay in time.

Defining localized functions

Definition 1

For $k \in \mathbb{N}$, let $\mathcal{X}^k(\mathbb{R})$ be the Banach space

$$\mathcal{X}^k(\mathbb{R}) := \{f \in L^\infty(\mathbb{R}) \mid f' \in H^{k-1}(\mathbb{R})\}$$

with norm

$$\|f\|_{\mathcal{X}^k(\mathbb{R})} := \|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{H^{k-1}(\mathbb{R})}.$$

- Denote $\langle x \rangle = (1 + x^2)^{1/2}$
- We then define a Banach space $\mathcal{X}_n^k(\mathbb{R})$ to be functions in $\mathcal{X}^k$ that approach their limits at infinity at a rate of $\langle x \rangle^n$.
- Similarly, we have $H_n^k(\mathbb{R})$ to be the Banach space with the weighted norm $\|f(x)\langle x \rangle^n\|_{H^k}$

Bounding interference term

Assumption 1

Assume that

$$f \in C([-\tau_0, \tau_0], \mathcal{X}_2^6(\mathbb{R})) \quad \textit{and} \quad g \in C([-\tau_0, \tau_0], H_2^6(\mathbb{R}))$$

for some $\tau_0 > 0$ *fixed. Furthermore, assume that* f *has fixed limits in its spatial variable at* $\pm\infty$ *given by* $f_{\pm\infty}$*.*

- This assumption says that $f - f_\infty$ and g remain localized for some amount of time.
- This implies that the interaction term $\phi(\cdot, \epsilon t)$ remains uniformly bounded in H^k for $t \in [-\epsilon^{-3}\tau_0, \epsilon^{-3}\tau_0]$

FPUT as first-order system

- We rewrite lattice equations as a first-order system

$$\begin{aligned} \dot{u}_n &= q_{n+1} - q_n, \\ \dot{q}_n &= u_n - u_{n-1} - \frac{1}{6}(u_n^3 - u_{n-1}^3), \end{aligned} \qquad n \in \mathbb{Z}.$$

- This is a Hamiltonian system where

$$\dot{U} = J\mathcal{H}'(U)$$

where $U = (u, q)$, J is the skew-symmetric, and $\mathcal{H}(U) = \sum_{n\in\mathbb{Z}} \frac{1}{2} q_n^2 + V(u_n)$.
- The fact that we have a natural choice for an energy function $\mathcal{H}(U)$ will be used later.

Introducing the error terms

- We assume solution to the first-order system is the ansatz plus some small error terms $\mathcal{U}(t)$ and $\mathcal{Q}(t)$

$$u_n(t) = \epsilon f(\epsilon(n+t), \epsilon^3 t) + \epsilon g(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \phi(\epsilon n, \epsilon t) + \mathcal{U}_n(t)$$
$$q_n(t) = \epsilon F(\epsilon(n+t), \epsilon^3 t) + \epsilon G(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \Phi(\epsilon n, \epsilon t) - \epsilon F_{-\infty} + \mathcal{Q}_n(t)$$

- The F, G, and Φ terms are chosen so that $\dot{u}_n(t) \approx q_{n+1} - q_n$ (ignoring the error terms) and so later residuals remain small.

Assumption 2

Assume that

$$\sum_{n=-\infty}^{\infty} \dot{u}_n(0) = \epsilon F_{+\infty} - \epsilon F_{-\infty}.$$

- The assumption and $-\epsilon F_{-\infty}$ terms are so that $\mathcal{Q} \in \ell^2$.

Equations for error terms

$$
\begin{aligned}
\dot{\mathcal{U}}_n(t) &= \mathcal{Q}_{n+1}(t) - \mathcal{Q}_n(t) + \mathrm{Res}_n^{(1)}(t) \\
\dot{\mathcal{Q}}_n(t) &= \mathcal{U}_n(t) - \mathcal{U}_{n-1}(t) \\
&\quad - \frac{1}{2}(\epsilon f(\epsilon(n+t)) + \epsilon g(\epsilon(n-ct)) + \epsilon^3\phi(\epsilon n))^2 \mathcal{U}_n(t) \\
&\quad + \frac{1}{2}(\epsilon f(\epsilon(n-1+t)) + \epsilon g(\epsilon(n-1-ct)) + \epsilon^3\phi(\epsilon(n-1)))^2 \mathcal{U}_{n-1}(t) \\
&\quad + \mathrm{Res}_n^{(2)}(t) + \mathcal{B}_n(\epsilon f + \epsilon g + \epsilon^3\phi, \mathcal{U})
\end{aligned}
\qquad n \in \mathbb{Z}
$$

- We get the following from plugging in our ansatz
- $\mathrm{Res}^{(1)}(t)$, $\mathrm{Res}^{(2)}(t)$ are residual terms
- $\mathcal{B}_n(\epsilon f + \epsilon g + \epsilon^3\phi, \mathcal{U})$ is a nonlinear term of $\mathcal{U}$.

Bounding residuals and non-linear terms

- Define

$$\delta = \max \left\{ \sup_{\tau \in [-\tau_0, \tau_0]} \|f(\cdot, \tau)\|_{\mathcal{X}_2^6}, \sup_{\tau \in [-\tau_0, \tau_0]} \|g(\cdot, \tau)\|_{H_2^6} \right\}$$

- From the choice of F, G, Φ we can get the bound

$$\|\mathrm{Res}^{(1)}(t)\|_{\ell^2} + \|\mathrm{Res}^{(2)}(t)\|_{\ell^2} \leq C\epsilon^{11/2}(\delta + \delta^5)$$

- We also have a bound on the nonlinear term

$$\|\mathcal{B}_n(\epsilon f + \epsilon g + \epsilon^3 \phi, \mathcal{U})\|_{\ell^2} \leq C\epsilon[(\delta + \epsilon^2 \delta^3)\|\mathcal{U}\|_{\ell^2}^2 + \|\mathcal{U}\|_{\ell^2}^3]$$

Energy function

- The equation for the error terms (ignoring the residuals and nonlinear terms) is essentially a non-autonomous Hamiltonian system. Thus we have the natural choice of an energy function

$$\mathcal{E}(t) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathcal{Q}_n^2(t) + \mathcal{U}_n^2(t) \\ - \frac{1}{2}\left(\epsilon f(\epsilon(n+t), \epsilon^3 t) + \epsilon g(\epsilon(n-ct, \epsilon^3 t) + \epsilon^3 \phi(\epsilon n, \epsilon t)\right)^2 \mathcal{U}_n^2(t)$$

- This energy bounds the error terms

$$\|\mathcal{Q}(t)\|_{\ell^2}^2 + \|\mathcal{U}(t)\|_{\ell^2}^2 \leq 4\mathcal{E}(t), \quad \text{for } t \in (-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}).$$

- And we have the following inequality

$$\left|\frac{d\mathcal{E}}{dt}\right| \leq C\mathcal{E}^{1/2}\left[\epsilon^{11/2}(\delta+\delta^5) + \epsilon^3\delta^2\mathcal{E}^{1/2} + \epsilon(\delta+\mathcal{E}^{1/2})\mathcal{E}\right]$$

Assumption 3

Suppose that the initial conditions for u satisfy

$$\|u(0) - \epsilon f(\epsilon\cdot, 0) - \epsilon g(\epsilon\cdot, 0)\|_{\ell^2} + \|\dot{u}(0) - \epsilon^2 \partial_1 f(\epsilon\cdot, 0) + \epsilon^2 \partial_1 g(\epsilon\cdot, 0)\|_{\ell^2_2} \leq \epsilon^{5/2}$$

and that $f(\cdot, 0) \in \mathcal{X}_2^6$ *and* $g(\cdot, 0) \in H_2^6$

- This assumption and the bound on $\left|\frac{d\mathcal{E}}{dt}\right|$ allow us to use a Grönwall type argument to get the following Theorem.

Theorem 2

Let assumption 1 hold and set

$$\delta = \max\left\{ \sup_{\tau\in[-\tau_0,\tau_0]} \|f(\cdot,\tau)\|_{\mathcal{X}_2^6}, \sup_{\tau\in[-\tau_0,\tau_0]} \|g(\cdot,\tau)\|_{H_2^6} \right\}$$

There exists positive constants ϵ_0 *and* C *such that for all* $\epsilon \in (0, \epsilon_0)$*, when initial data* $(u(0), \dot{u}(0))$ *satisfy assumptions 2 and 3, the unique solution* (u, q) *to the FPU equation section 3 belongs to*

$$C^1([-t_0(\epsilon), t_0(\epsilon)], \ell^\infty(\mathbb{Z}))$$

with $t_0(\epsilon) := \epsilon^{-3}\tau_0$ *and satisfies*

$$\begin{aligned} &\|u(t) - \epsilon f(\epsilon(\cdot + t), \epsilon^3 t) - \epsilon g(\epsilon(\cdot - ct), \epsilon^3 t)\|_{\ell^2} \\ &\quad + \|\dot{u}(t) - \epsilon\partial_1 f(\epsilon(\cdot + t), \epsilon^3 t) + \epsilon^2\partial_1 g(\epsilon(\cdot - ct), \epsilon^3 t)\|_{\ell^2} \leq C\epsilon^{5/2}, \end{aligned}$$

for $t \in [-t_0(\epsilon), t_0(\epsilon)]$.

*Under certain assumptions this can be extended to $\epsilon^{-3}|log(\epsilon)|$ time.

Stability

Stability of the kink-like solution

- There are several notions of stability for traveling wave solutions
  - Linear stability
  - Orbital stability
  - Asymptotic stability
- We will focus only on the weakest form of stability, which is linear stability with the goal to eventually show asymptotic stability.
- Idea: first demonstrate linear stability of kink solution to mKdV then use this to prove linear stability of the kink-like solution to the FPUT.

Spectrum of linearization around φ_c

- We linearize around the kink solution φ_c, changing to a co-moving frame.
  - The essential spectrum is $i\mathbb{R}$, so we don't have spectral stability
- Redefining on an exponentially weighted space L^2_a gives a new operator A_a. The essential spectrum is given by

$$S_e^a = \{-(ik - a)^3 + 2c(ik - a) \mid k \in \mathbb{R}\},$$

- The point spectrum to the right of S_e^a is found using the Evans function for A_a.
- Only eigenvalue to the right of S_e^a is $\lambda = 0$, which is simple.

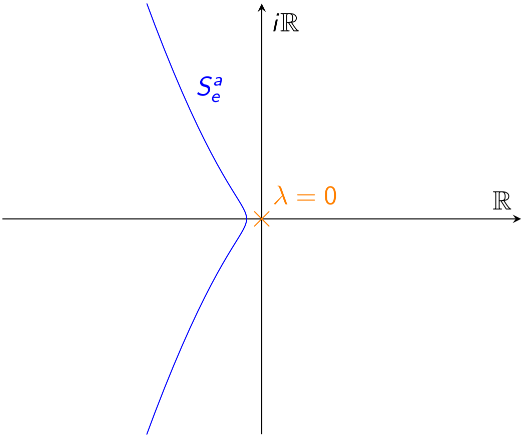

Figure: The spectrum of A_a for $0 < a < \sqrt{2c/3}$. The essential spectrum S_e^a is contained in the left-hand side of $\mathbb{C}$. The only eigenvalue to the right of S_e^a is the simple eigenvalue $\lambda = 0$.

Smoothing and decay results

- If we remove the zero eigenvalue using a projection operator, then we will have the remainder of the spectrum bounded away from $i\mathbb{R}$ and in the left-half plane of $\mathbb{C}$.
- Thus we can show the following result, which gives us smoothing and decay in the semigroup generated by A_a.

Theorem 3

Assume that $0 < a < \sqrt{2c/3}$ *and that the spectral projection for* A_a *associated with* $\lambda = 0$ *is given by* P*. Let* $I - P = Q$*. Then* A_a *is the generator of a* C^0 *semigroup on* H^s *for any real* s*, and, for any* $b > 0$ *such that the* L^2*-spectrum* $\sigma(A_a) \subset \{\lambda \mid \mathrm{Re}(\lambda) < -b\} \cup \{0\}$*, there exists* C *such that for all* $w \in L^2$ *and* $t > 0$*,*

$$\|e^{A_a t} Q w\|_{H^1} \leq C t^{-1/2} e^{-bt} \|w\|_{L^2}.$$

Sketch of the proof of linear stability of kink-like solution

- In [Miz13], the asymptotic stability of N-solitary wave solutions of the FPUT was shown. We follow a similar argument.
- We have an approximate solution to the FPUT where

$$u_\epsilon(t,n) = \begin{pmatrix} r_\epsilon(t,n) \\ -r_\epsilon(t,n) \end{pmatrix}.$$

  and $r_\epsilon(t,x) = \epsilon\varphi_1(\epsilon(x - c_\epsilon t))$
- If $u_\epsilon(t) + \gamma(t)$ is the kink solution, then the linearized equation is

$$\partial_t w(t) = JH''(u_\epsilon(t) + \zeta(t))w(t)$$

- Goal: show for every $t > s \geq 0$

$$\|e^{\epsilon a(\cdot - c_\epsilon t)} w(t)\|_{\ell^2} \leq Me^{-b\epsilon^3(t-s)}\|e^{\epsilon a(\cdot - c_\epsilon s)} w(s)\|_{\ell^2}.$$

  given some orthogonality conditions on $w(t)$.

Sketch of the proof of linear stability of kink-like solution

1. Take a discrete Fourier transform of the linear equation.
2. Break the problem up into low-frequency, mid-frequency, high-frequency parts of the Fourier transform.
3. The mid-frequency and high-frequency parts can be controlled directly, while the low-frequency part is controlled using the decay estimates of the linearization around the kink solutions of the mKdV.

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Questions?